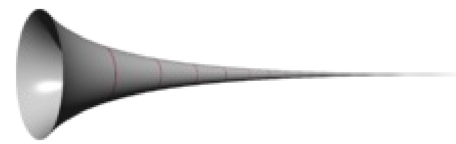
Gabriel's Horn



3D illustration of Gabriel's horn.

Gabriel's horn (also called **Torricelli's trumpet**) is a particular geometric figure that has infinite surface area but finite volume. The name refers to the Christian tradition identifying the archangel Gabriel as the angel who blows the horn to announce Judgment Day, associating the divine, or infinite, with the finite. The properties of this figure were first studied by Italian physicist and mathematician Evangelista Torricelli in the 17th century.

Mathematical definition

Gabriel's horn is formed by taking the graph of

$$x\mapsto {\frac {1}{x}},$$

with the domain $x\geq 1$ and rotating it in three dimensions about the x-axis. The discovery was made using Cavalieri's principle before the invention of calculus, but today calculus can be used to calculate the volume and surface area of the horn between $x = 1$ and $x = a$, where $a > 1$. Using integration (see Solid of revolution and Surface of revolution for details), it is possible to find the volume V and the surface area A:

$$V=\pi \int \limits _{1}^{a}\left({\frac {1}{x}}\right)^{2}\mathrm {d} x=\pi \left(1-{\frac {1}{a}}\right)$$

$$A=2\pi \int \limits _{1}^{a}{\frac {1}{x}}{\sqrt {1+\left(-{\frac {1}{x^{2}}}\right)^{2}}}\mathrm {d} x>2\pi \int \limits _{1}^{a}{\frac {\mathrm {d} x}{x}}=2\pi \ln(a).$$

The value a can be as large as required, but it can be seen from the equation that the volume of the part of the horn between $x = 1$ and $x = a$ will never exceed π; however, it does gradually draw nearer to π as a increases. Mathematically, the volume *approaches* π as a *approaches* infinity. Using the limit notation of calculus:

$$\lim _{a\to \infty }V=\lim _{a\to \infty }\pi \left(1-{\frac {1}{a}}\right)=\pi \cdot \lim _{a\to \infty }\left(1-{\frac {1}{a}}\right)=\pi .$$

The surface area formula above gives a lower bound for the area as 2π times the natural logarithm of a. There is no upper bound for the natural logarithm of a, as a approaches infinity. That means, in this case, that the horn has an infinite surface area. That is to say,

$$\lim _{a\to \infty }A\geq \lim _{a\to \infty }2\pi \ln(a)=\infty .$$

Apparent paradox

When the properties of Gabriel's horn were discovered, the fact that the rotation of an infinitely large section of the xy-plane about the x-axis generates an object of finite volume was considered a paradox. While the section lying in the xy-plane has an infinite area, any other section parallel to it has a finite area. Thus the volume, being calculated from the "weighted sum" of sections, is finite.

Another approach is to treat the horn as a stack of disks with diminishing radii. The sum of the radii produces a harmonic series that goes to infinity. However, the correct calculation is the sum of their squares. Every disk has a radius $r = 1/x$ and an area πr^2 or π/x^2. The series $1/x$ diverges but $1/x^2$ converges. In general, for any real $\varepsilon > 0$, $1/x^{1+\varepsilon}$ converges.

The apparent paradox formed part of a dispute over the nature of infinity involving many of the key thinkers of the time including Thomas Hobbes, John Wallis and Galileo Galilei.[1]

There is a similar phenomenon which applies to lengths and areas in the plane. The area between the curves $1/x^2$ and $-1/x^2$ from 1 to infinity is finite, but the lengths of the two curves are clearly infinite.

Painter's paradox

Since the horn has finite volume but infinite surface area, there is an apparent paradox that the horn could be filled with a finite quantity of paint and yet that paint would not be sufficient to coat its inner surface. The paradox is resolved by realizing that a finite amount of paint can in fact coat an infinite surface area — it simply needs to get thinner at a fast enough rate. (Much like the series $1/2^N$ gets smaller fast enough that its sum is finite.) In the case where the horn is filled with paint, this thinning is accomplished by the increasing reduction in diameter of the throat of the horn.

Converse

The converse of Gabriel's horn—a surface of revolution that has a *finite* surface area but an *infinite* volume—cannot occur when revolving a continuous function on a closed set:

Theorem

Let $f : [1,\infty) \to [0,\infty)$ be a continuously differentiable function. Write S for the solid of revolution of the graph $y = f(x)$ about the x-axis. *If the surface area of S is finite, then so is the volume.*

Proof

Since the lateral surface area A is finite, the limit superior:

$$\begin{aligned}\lim _{t\to \infty }\sup _{x\geq t}f(x)^{2}~-~f(1)^{2}&=\limsup _{t\to \infty }\int \limits _{1}^{t}\left(f(x)^{2}\right)'\mathrm {d} x\\&\leq \int \limits _{1}^{\infty }\left|\left(f(x)^{2}\right)'\right|\mathrm {d} x=\int \limits _{1}^{\infty }2f(x)\left|f'(x)\right|\mathrm {d} x\\&\leq \int \limits _{1}^{\infty }2f(x){\sqrt {1+f'(x)^{2}}}\mathrm {d} x={\frac {A}{\pi }}\\&<\infty .\end{aligned}$$

Therefore, there exists a t_0 such that the supremum $\sup\{ f(x) \mid x \geq t_0\}$ is finite. Hence,

$M = \sup\{ f(x) \mid x \geq 1\}$ must be finite since f is a continuous function, which implies that f is bounded on the interval $[1,\infty)$.

Finally, the volume:

$$\begin{aligned}V&=\int \limits _{1}^{\infty }f(x)\cdot \pi f(x)\mathrm {d} x\\&\leq \int \limits _{1}^{\infty }{\frac {M}{2}}\cdot 2\pi f(x)\mathrm {d} x\\&\leq {\frac {M}{2}}\cdot \int \limits _{1}^{\infty }2\pi f(x){\sqrt {1+f'(x)^{2}}}\mathrm {d} x={\frac {M}{2}}\cdot A.\end{aligned}$$

Therefore: *if the area A is finite, then the volume V must also be finite.*

See also

References

1. ^ *Havil, Julian (2007). Nonplussed!: mathematical proof of implausible ideas. Princeton University Press. pp. 82–91. ISBN 0-691-12056-0.*

Further reading

- *Fleron, Julian F. "Gabriel's Wedding Cake" (PDF).*
- *Lynch, Mark. "A Paradoxical Paint Pail".*

External links

- Torricelli's trumpet at PlanetMath
- "Gabriel's Horn" by John Snyder, the Wolfram Demonstrations Project, 2007.
- Gabriel's Horn: An Understanding of a Solid with Finite Volume and Infinite Surface Area by Jean S. Joseph.