Traveling Waves on Transmission Systems

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Synopsis.—*The purpose of this paper is two-fold: First; to present the theory of traveling waves on multi-conductor systems, and second, to compile a brief compendium on the general subject of traveling waves on transmission systems. While the application of the multi-conductor theory is more laborious than that of the single-wire theory, yet it does not involve much greater complication, and it becomes necessary to go to the more general theory when mutual effects are important, as in the study of ground wires, or when discontinuities exist in paralleled circuits carrying traveling waves. The origin, shape, and general characteristics of traveling waves are discussed. The equivalent circuits of terminal equipment and the corresponding reflections and refractions from such junctions are given for a large number of cases. The methods of computing a multiplicity of successive reflections by means of lattices are described. The effect of line losses in equalizing the subsidence of traveling waves and on their attenuation and distortion is also discussed.*

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ORIGIN OF TRAVELING WAVES

SURGES on transmission lines owe their origin to four different causes; induced from lightning discharges, direct lightning stroke, arcing grounds, and switching.

The maximum potential at the point of origin, and the crest of induced traveling waves are respectively:

$$V = \alpha G h$$
$$V' = \alpha' G h$$

where h is the height of the line conductor, G the field gradient, and α and α' factors depending on the law of cloud discharge and the distribution of bound charge.[1] For an instantaneous cloud discharge (an impossible assumption) $\alpha = 1.0$ and $\alpha' = 0.5$. (Fig. 1) As the time of cloud discharge increases both α and α' decrease, and rapidly approach equality; so that for a cloud discharge slower than 10 microseconds they are practically equal. The equation for α' based on the assumption of an exponential law of cloud discharge is:

$$\alpha' = \frac{1}{2}\left(1 - \epsilon^{-\frac{3x}{t}}\right)$$

where x is the length of the bound charge in thousands of feet, and t is the time of cloud discharge in microseconds. The length of the wave is:

$$L = x + t$$

and the front of the wave is x or t depending upon which is the smaller. It is evident that induced strokes become harmless for long periods of cloud discharge, and that high potential induced strokes are possible only with very short waves. For the accepted maximum value of $G = 100$ kv/ft., an average line height of $h = 60$ ft., a 3,000-ft. rectangular bound charge, and 10 microseconds for the cloud discharge, the maximum induced traveling wave that can occur on the line is:

$$V' = \alpha' G h = 0.30 \times 100 \times 60 = 1{,}800 \text{ kv.}$$

and this wave has a total length of only 13 microseconds. Had the time of cloud discharge been 30 microseconds then the induced voltage would have been only

$$V' = 0.13 \times 100 \times 60 = 780 \text{ kv.}$$

The induced voltages decrease in importance as the line insulation is increased, and the proportion of outages due to direct strokes increases. The induced voltages can be roughly halved by the use of one or more ground wires.

The most feasible way of correlating the direct versus induced stroke arguments is through the simple theoretical relationships between the traveling waves originating from these two causes. It was found in a previous investigation[1] that the law of cloud discharge has a greater effect on the shape of the traveling wave than the distribution of bound charge, and therefore that the assumption of a rectangular bound charge may be made without involving any marked departure from the true shape of the traveling wave. Then the determination of the wave shape reduces to the problem

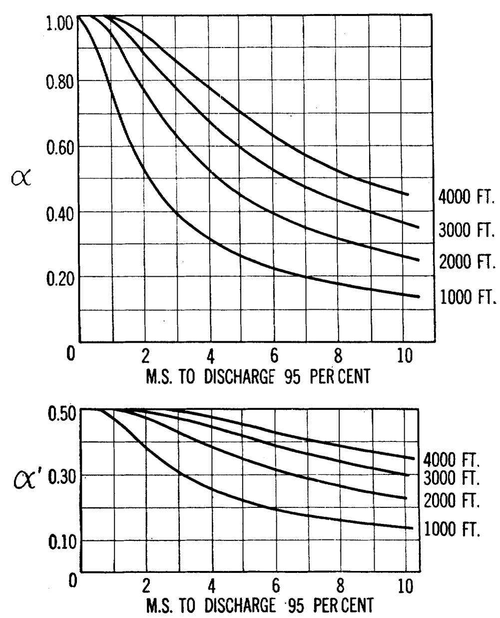

FIG. 1—REDUCTION FACTORS FOR INDUCED LIGHTNING POTENTIALS

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1. For references see Bibliography.

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of finding the law of cloud discharge. If the wave measured by the cathode-ray oscillograph station was due to a direct stroke, then the shape of the current wave in the lightning bolt is known. But if Q is the charge on the cloud, this current is

$$i = \frac{\partial Q}{\partial t} = Q_o \frac{\partial}{\partial t} F(t)$$

where Q_o is the total charge at the beginning of cloud discharge, and $F(t)$ is the law of cloud discharge. Therefore

$$F(t) = \frac{1}{Q_o} \int_0^t i \, dt$$

The actual numerical values of Q_o and i are unimportant. All that is necessary is to arbitrarily choose

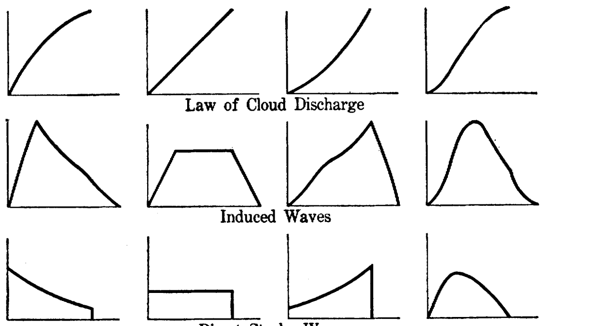

FIG. 2—INDUCED AND DIRECT STROKE WAVES CORRESPONDING TO DIFFERENT LAWS OF CLOUD DISCHARGE

Q_o of such value that $F(t)$ reaches a final value of unity at the completion of cloud discharge. And the cloud discharge is complete when i has ceased. Suppose that the direct stroke wave has the typical shape, Fig. 3D,

$$i = I(\epsilon^{-at} - \epsilon^{-bt})$$

Then the law of cloud discharge is

$$F(t) = \frac{I}{Q_o} \int_0^t (\epsilon^{-at} - \epsilon^{-bt}) \, dt$$

$$= \frac{I}{Q_o} \left\{ \frac{1 - \epsilon^{-at}}{a} - \frac{1 - \epsilon^{-bt}}{b} \right\}$$

In Fig. 2 are shown a few simple direct stroke waves, and the corresponding laws of cloud discharge and induced waves. Thus if a cathode-ray oscillogram is definitely known to be that of either a direct or induced stroke, then the character of the other may be derived. Of course the length and shape of the bound charge distribution will influence the shape of the induced traveling wave, but not nearly to the same extent as the law of cloud discharge. The above expressions follow from the premise that the electrostatic field of the cloud collapses *uniformly*. Possibly there is considerable departure from this assumption. Nevertheless, the above equations do correlate the two types of lightning waves as near as our present knowledge of the mechanism of cloud discharge will permit.

The third cause of abnormal voltage oscillations on a transmission line is arcing grounds.[7] Theoretically, voltages as high as 7½ times normal line-to-neutral voltage are possible. But many assumptions underlie the present theories of arcing grounds. When the results arrived at by theory are judiciously shaded to compensate for the assumptions introduced to simplify the analysis, it is doubtful if an arcing ground can actually cause a voltage in excess of 5 times normal line-to-neutral voltage on an isolated neutral system. Arcing ground surges are oscillatory in nature, and consist of a normal frequency oscillation which gradually builds up to several times normal, and a superimposed high frequency oscillation, whose frequency depends chiefly upon the length of line between the station and the arcing ground.

Switching surges exhibit the same class of characteristics as arcing grounds, although they are more irregular in shape. Their proper consideration is outside the scope of this paper.

SHAPES AND SPECIFICATIONS OF TRAVELING WAVES

The principal shapes of most natural waves are included in Fig. 3, that is, may be represented by the difference of two exponentials, but of course actual waves are usually serrated by minor irregularities.

As far as mathematical simplicity is concerned the

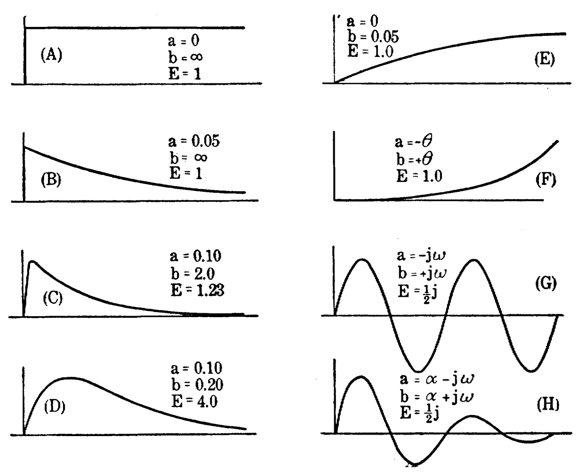

FIG. 3—EMPIRICAL WAVE SHAPES GIVEN BY $e = E(\epsilon^{-at} - \epsilon^{-bt})$.

most simple wave to calculate the effects of, is the infinite rectangular, shown in Fig. 3A. Also, as a rule, such a wave is the most dangerous to terminal equipment, and therefore calculations based on it are apt to err on the side of safety. Still other reasons that have favored its use in analysis are that it is by far the easiest to study pictorially, and that until recently the actual shapes of lightning surges were not known. However, during the past few years a great many cathode-ray oscillograms of natural lightning waves have been obtained under many different conditions, so that fairly definite information as to their general shape and characteristics is available. It is therefore essential that calculations be made with these characteristic lightning waves, in order that the influence of the fronts, tails, and lengths of the waves may be evaluated.

Three different methods for calculating waves of arbitrary shape are given in the following:

1. Express in operational notation, combine with the function representing the reflection or refraction operator, and solve the resulting operational equations.
2. Consider the wave as made up of a series of infinite rectangular waves, and add up the solutions corresponding to each component rectangular wave. This superposition may be done graphically as shown in Fig. 5 or mathematically by means of Duhamel's theorem.
3. Express the wave as the sum of a number of functions for which the individual solutions are known or can be found, and add these solutions. Fig. 4 illustrates a few simple examples.

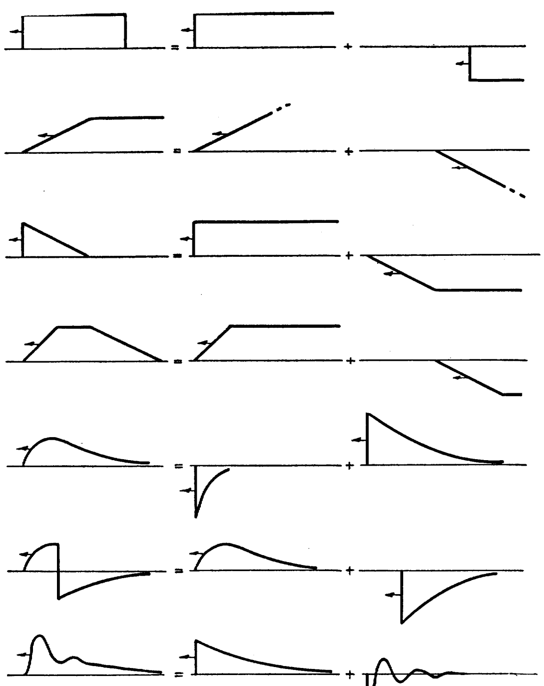

FIG. 4—COMPOUNDING OF SIMPLE WAVES TO OBTAIN COMPLEX WAVES

Which of these methods to use is entirely a matter of convenience in any specific case, each method having certain advantages and limitations. The difficulty usually encountered with the first method is that it complicates the operational equations so that a solution either cannot be obtained at all, or else only by the most laborious and complex process. As a rule, in dealing with the behavior of traveling waves at a transition point, it will be found that the reflection (or other) operator acting on the simple unit function (an infinite rectangular wave) is just about as complicated a proposition as can be handled with engineering expedience, and that any further complications in the operational equations are prohibitive of a solution. The application of Duhamel's theorem suffers from substantially the same defects, except that the trouble with it is in making a difficult integration. It is

$$f(t) = E(o) \cdot \phi(t) + \int_0^t \phi(\tau) \frac{\partial}{\partial t} E(t - \tau)\, d\tau$$

where

$\phi(t)$ = solution corresponding to unit function

$E(t)$ = applied wave of arbitrary shape

$f(t)$ = solution corresponding to $E(t)$

The graphical representation of a wave of arbitrary shape as a set of rectangular components, is of course only an approximation, but in a great many cases of engineering importance it is quite sufficient, and has the advantage of simplicity. In any event it can always be made as accurate as required, merely by subdividing the wave into a sufficiently large number of small rectangular components. In many cases the incident wave may be so complicated as to defy analytic expression, and then a graphical break-up into rectangular components is the only way out of the difficulty.

The third method, that of representing the wave as a sum of functions for which the solutions are known, is very powerful and practicable. There are a few simple functions for which the response of a network can usually be computed with reasonable ease, and by compounding such functions almost any desired wave shape can be reproduced to a good approximation. These elementary waves are:

a. Infinite rectangular
b. Simple exponential
c. Uniformly rising front
d. Damped sinusoid
e. Difference of two exponentials

As a matter of fact, by a suitable choice of the parameters in the difference of two exponentials, all of these elementary waves may be considered as special cases of *e*, as illustrated in Fig. 3. It so happens that the solution corresponding to

$$e_1 = E_1 \epsilon^{-at}$$

is easily effected by means of Heaviside's shifting theorem.

$$f(p)\, \epsilon^{-at} = \epsilon^{-at} f(p - a)$$

and in many cases with no greater mathematical difficulty than attains with the unit function alone.

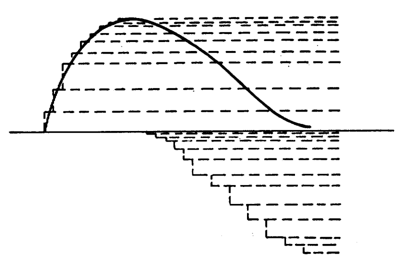

FIG. 5—APPROXIMATION BY RECTANGULAR COMPONENTS

The references should be consulted for more details concerning the specification of traveling waves.[1,2,6]

Attenuation of Traveling Waves

It is usually justifiable to calculate traveling waves on the assumption of no losses, and then to compensate for the attenuation by an exponential decrement factor, arrived at experimentally.

Corona is the chief factor in causing attenuation and distortion. It levels off the top and elongates the wave, but its effect varies with the weather and other conditions, so that there is no such thing as a definite attenuation on a given line.

The effect of the line losses on traveling waves are three-fold: (1), the waves of voltage and current are attenuated, (2), the shapes are distorted with time, (3), the current and voltage waves depart from exact similarity, so that they are no longer connected by the simple linear proportionality factors called the surge impedances.

Fig. 8 shows the effect of attenuation in the charging of an open-ended line from an infinite voltage source. Without losses, the cycle of oscillations repeats indefinitely, but when line losses are present the oscillations gradually diminish until the line eventually reaches a steady state condition. However, a distortionless line can never become fully charged to the terminal potential, throughout its length, for the distortionless feature requires the presence of both leakage conductance and series resistance. The flow of current in the leakage conductance results in a progressive voltage drop along the line. Therefore, the ultimate level charging of a line requires that there be no leakage currents. Referring to Fig. 8, which has been drawn for a linear rather than exponential attenuation, the voltage at various instances of time for an attenuation $(1-\alpha)$ per trip is

Sending end	Units of time	Receiving end
1	0	0
1	1	α
$1+\alpha^2$	2	2α
1	3	$2\alpha - \alpha^3$
$1-\alpha^4$	4	$2\alpha - 2\alpha^3$
1	5	$2\alpha - 2\alpha^3 + \alpha^5$
$1+\alpha^6$	6	$2\alpha - 2\alpha^3 + 2\alpha^5$
1	7	$2\alpha - 2\alpha^3 + 2\alpha^5 - \alpha^7$
$1-\alpha^8$	8	$2\alpha - 2\alpha^3 + 2\alpha^5 - 2\alpha^7$
etc.		etc.

The voltage at the receiving end after $4n$ units of time is

$$e = 2(\alpha - \alpha^3 + \alpha^5 - \alpha^7 + \ldots\ldots)$$
$$= 2(\alpha - \alpha^3)(1 + \alpha^4 + \alpha^8 + \ldots\ldots)$$
$$= 2(\alpha)(1-\alpha^2)\sum_{o}^{n}\alpha^{4r} = 2\alpha\frac{1-\alpha^{4(n+1)}}{1+\alpha^2}$$

After an infinite number of oscillations

$$e = \frac{2\alpha}{1+\alpha^2}$$

Thus if the attenuation is $(1-\alpha) = 0.5$ as in Fig. 8, and the line is distortionless, the open end of the line finally stabilizes at a voltage of

$$e = \frac{2\times 0.5}{1+0.25} = 0.8$$

General Properties of Free Traveling Waves

In Appendix I the general differential equations for potentials and currents of a multi-conductor transmission line, (Fig. 6), are formulated. Each conductor, (Fig. 7), is assumed to have its own resistance to ground

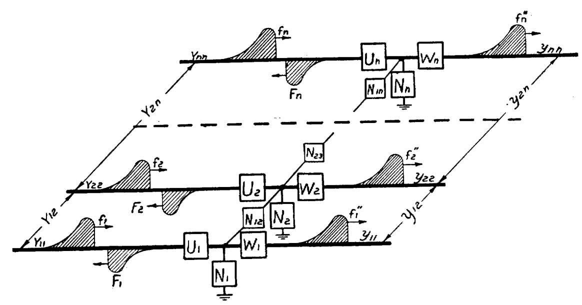

Fig. 6—General Multi-Conductor System

and to each of the other conductors, self and mutual inductance between all wires, and leakage conductance from each wire to earth and to all the other wires. For an n-wire system there results a set of n simultaneous partial differential equations of the second order in both the time and distance derivatives. When these n equations are solved for any potential there results a linear partial differential equation with constant coefficients of the $2n$ order in both the time and distance derivatives. This general differential equation is given as a determinate whose formal expansion, according to algebraic rules, yields the polynominal form. The

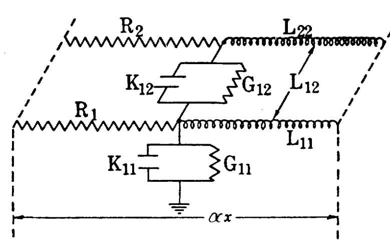

Fig. 7—Circuit Constants of Mutually Coupled Circuits

differential equation and its coefficients is identical for the potential on any wire of the n-wire system; but the boundary conditions, and therefore the integration constants of the solution, may be different for each wire. The general linear partial differential equation with constant coefficients has a formal solution, but it has not been included in this paper (see "Treatise on Differential Equations" by Forsyth). But the solutions are given corresponding to the following special cases:

The No-Loss Line
The Completely Transposed Line
Solution for Alternating Currents

If a line is free from losses the differential equations of the system are satisfied by pure wave functions. These waves travel without attenuation or distortion until a transition point on the circuit is reached, where they reflect and refract in accordance with the general laws given in Appendix II. In an n-wire no-loss system there are n possible velocities of wave propagation, but in the case of overhead conductors these n velocities all become equal to the velocity of light.

If the line is completely transposed, so that every conductor occupies the same relative position as every other conductor, and for the same distance, then the possible average velocities between terminals are reduced to two in number. If the line is made up of g groups, and the conductors of each group are completely transposed with respect to their own group, then there are $(g + 1)$ possible average velocities.

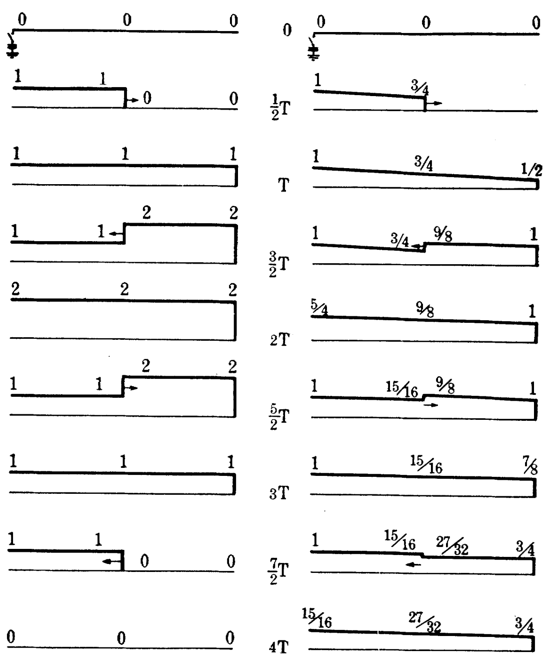

FIG. 8—EFFECT OF ATTENUATION ON CHARGING A LINE

The conventional method of calculating the steady state alternating-current behavior of a transmission line by using "constants to neutral" is a very special case, but is rigorous for a single-circuit, three-phase, completely transposed line. To obtain these simplified equations directly from the general alternating current solution is a matter of considerable manipulation. The reduction is carried out in detail in Appendix I.

In Appendix I it is shown that in an n-wire circuit the energy of free traveling waves is equally divided between the electrostatic and the electromagnetic fields. However, when these waves reach a transition point, or when waves traveling in opposite directions pass through one another, the energy balance is upset. Whether the energy resides mostly in one field or the other during a transition period depends entirely upon the nature of the transition.

Pure traveling waves of voltage and current on an n-wire system, are related to each other by a set of simultaneous linear equations having constant coefficients called the self and mutual surge impedances of the line. These equations[3] are

$$\begin{aligned} \pm e_1 &= z_{11} i_1 + z_{21} i_2 + \dots \dots + z_{n1} i_n \\ \pm e_2 &= z_{12} i_1 + z_{22} i_2 + \dots \dots + z_{n2} i_n \\ &\dots \\ \pm e_n &= z_{1n} i_1 + z_{2n} i_2 + \dots \dots + z_{nn} i_n \end{aligned}$$

where the (+) sign is used for waves traveling in the forward direction, and the (−) sign for waves moving in the reverse direction. Also

$$z_{rr} = 60 \log \left(\frac{2h}{\rho} \right) = \text{self surge impedance of conductor } r$$

$$z_{rs} = 60 \log \left(\frac{a}{b} \right) = \text{mutual surge impedance between } r \text{ and } s$$

h = height of r above ground plane
a = distance between r and image of s
b = distance between r and s
ρ = radius of conductor

These equations, in conjunction with the principle of superposition and the conditions of voltage and current continuity at a junction or transition point, define the behavior of pure traveling waves on transmission systems. The principle of superposition is involved in calculating the effect of more than one independent wave at the same point simultaneously, such as occur for successive reflections, meeting of circuits, etc.

Instead of writing the voltages in terms of the currents and the surge impedances, the arrangement may be reversed and the currents written in terms of the voltages and surge admittances as follows:

$$\begin{aligned} \pm i_1 &= y_{11} e_1 + y_{21} e_2 + \dots \dots + y_{n1} e_n \\ \pm i_2 &= y_{12} e_1 + y_{22} e_2 + \dots \dots + y_{n2} e_n \\ &\dots \\ \pm i_n &= y_{1n} e_1 + y_{2n} e_2 + \dots \dots + y_{nn} e_n \end{aligned}$$

If

$$D = \begin{vmatrix} z_{11} & z_{21} & \dots & z_{n1} \\ z_{12} & z_{22} & \dots & z_{n2} \\ \dots & & & \\ z_{1n} & z_{2n} & \dots & z_{nn} \end{vmatrix}$$

and

$$D' = \begin{vmatrix} y_{11} & y_{21} & \dots & y_{n1} \\ y_{12} & y_{22} & \dots & y_{n2} \\ \dots & & & \\ y_{1n} & y_{2n} & \dots & y_{nn} \end{vmatrix}$$

then the y's and z's are related to each other as

$$y_{rs} = \frac{(\text{minor of } D \text{ for which the cofactor is } z_{rs})}{D}$$

$$z_{rs} = \frac{(\text{minor of } D' \text{ for which the cofactor is } y_{rs})}{D'}$$

For instance, in the case of two conductors

$$z_{12} = \frac{y_{21}}{\begin{vmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{vmatrix}} = \frac{-\,y_{21}}{y_{11}\,y_{22} - y_{12}\,y_{21}}$$

$$z_{11} = \frac{y_{22}}{\begin{vmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{vmatrix}} = \frac{y_{22}}{y_{11}\,y_{22} - y_{12}\,y_{21}}$$

Traveling Waves on a Single-Conductor Circuit

In the majority of problems dealing with traveling waves, it is sufficient to make the calculations on the basis of a single-wire circuit. The return part of the circuit may be either a second conductor or the ground.

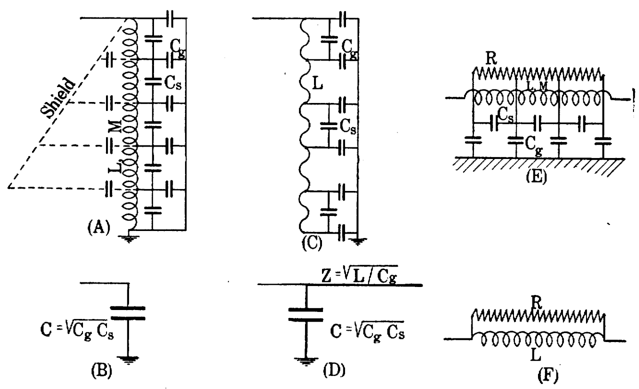

FIG. 9—EQUIVALENT CIRCUITS OF TERMINAL APPARATUS TO LIGHTNING SURGES

(A) Transformer, ideal
(B) Transformer, approx.
(C) Rotating machine, ideal
(D) Rotating machine, approx.
(E) Reactor with shunt resistor, ideal
(F) Reactor with shunt resistor, approx.

When the single-circuit theory is applicable, the potential and current waves are proportional to each other by the surge impedance of the circuit;

$e\,(x - v\,t) = Z\,i\,(x - v\,t)$ for forward moving waves

$e\,(x + v\,t) = -\,Z\,i\,(x + v\,t)$ for backward moving waves

where

$Z = \sqrt{L/C}$ = surge impedance in ohms
$Y = 1/Z$ = surge admittance in mhos
L = inductance in henrys
C = capacitance in farads

It is to be noticed that for waves traveling in the positive or forward direction that e and i have the same sign; but that for waves traveling in the reverse or negative direction they have opposite signs. However, it is immaterial which direction along the line be chosen as positive or forward; provided that the positive sense be strictly adhered to throughout the calculations.

The reflection and refraction operators, by equations (**18**) and (**19**) of Appendix II are respectively

$$\frac{Z_o - Z}{Z_o + Z} \text{ and } \left(\frac{1}{1 + N\,(W + z)}\right)\left(\frac{2z}{Z_o + Z}\right)$$

where Z_o is the total impedance, expressed in operational form, at the transition point as viewed from the approaching incident wave, z is the surge impedance of the line, and N and W are impedances having the same meaning as in Fig. 6. When these operators are applied to any incident wave e, expressed as a function of time t counted from the instant that e arrives at the transition point, they derive the reflected and transmitted waves e' and e'' respectively.

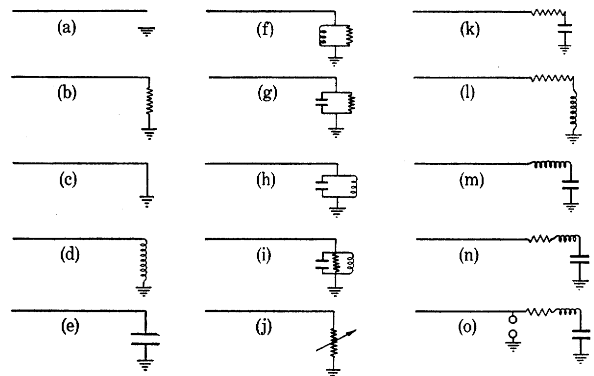

FIG. 10—TERMINAL CONDITIONS ON SINGLE-CONDUCTOR CIRCUITS

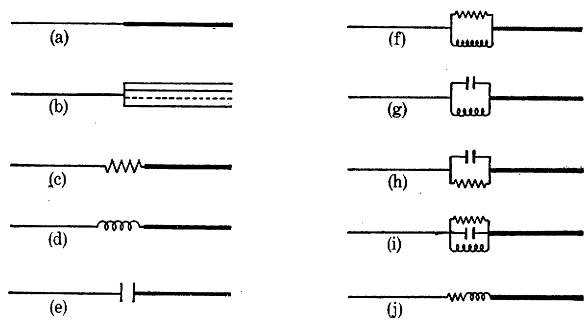

FIG. 11—JUNCTIONS BETWEEN SINGLE-CONDUCTOR CIRCUITS

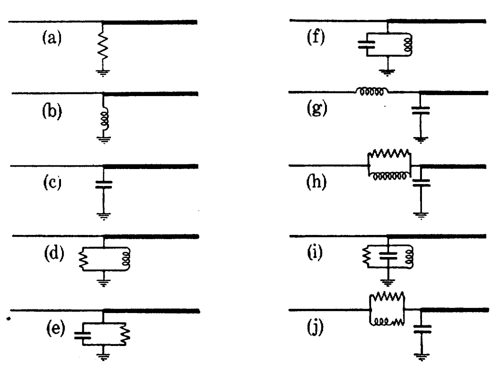

FIG. 12—JUNCTIONS BETWEEN SINGLE-CONDUCTOR CIRCUITS

The solutions and graphs of most of the thirty-five circuits shown in Figs. 10, 11, and 12 have been given in previous papers but the equations will be repeated here for ready reference. By a fortunate coincidence, nearly all of these circuits are included by two general equations, through an adjustment of the coefficients thereof. Therefore, all that is necessary here is to give these equations, and a table with the proper coefficients for each case. The equations are

$$A \left[\frac{a + \alpha}{a - \beta} \epsilon^{-at} - \frac{\alpha + \beta}{a - \beta} \epsilon^{-\beta t} \right] E \qquad (1)$$

$$A \left[\frac{\omega_o^2 - 2\, a\, \alpha + a^2}{\omega_o^2 - 2\, a\, \beta + a^2} \epsilon^{-at} + \frac{2\,(\alpha - \beta)\, \epsilon^{-\beta t}}{\omega\,(\omega_o^2 - 2\, a\, \beta + a^2)} \{(\omega_o^2 - a\,\beta) \sin \omega t + a\, \omega \cos \omega t\} \right] E \qquad (2)$$

$$\omega_o^2 = 1/L\,C \text{ and } \omega^2 = (\omega_o^2 - \beta^2),\ e = E\,\epsilon^{-at}$$

In the following tables the reflected wave e' and the transmitted wave e'' are given. In all cases the total voltage at the transition point is

$$e_o = e + e'$$

The reflected and transmitted current waves are respectively

$$i' = -\frac{e'}{z_1} \text{ and } i'' = \frac{e''}{z_2}$$

and the total current at the transition point is

$$i_o = i + i'$$

TABLE I

Fig.	Equation	α	β	A
10-a (1)*	$e' = e$			
10-b (1)	$e' = \frac{R - Z}{R + Z} e$			
10-c (1)	$e' = -e$			
10-d (1)	e' = (I.)	$\frac{Z}{L}$	$\frac{Z}{L}$	1
10-e (1)	e' = (I.)	$\frac{1}{C\,Z}$	$\frac{1}{C\,Z}$	-1
10-f (1)	e' = (I.)	$\frac{Z\,R}{L\,(R - Z)}$	$\frac{Z\,R}{L\,(R + Z)}$	$\frac{R - Z}{R + Z}$
10-g (1)	e' = (I.)	$\frac{R - Z}{Z\,R\,C}$	$\frac{R + Z}{Z\,R\,C}$	-1
10-h (1)	e' = (II.)	$-\frac{1}{2\,C\,Z}$	$\frac{1}{2\,C\,Z}$	-1
10-i (1)	e' = (II.)	$\frac{Z - R}{2\,R\,C\,Z}$	$\frac{Z + R}{2\,R\,C\,Z}$	-1
10-j (10)	$e' = e - z\,i_0$			
10-k (6)	e' = (I.)	$\frac{1}{C\,(Z - R)}$	$\frac{1}{C\,(Z + R)}$	$\frac{R - Z}{R + Z}$
10-l	e' = (I.)	$\frac{Z - R}{L}$	$\frac{Z + R}{L}$	1
10-m (5)	e' = (II.)	$\frac{-Z}{2\,L}$	$\frac{Z}{2\,L}$	1
10-n (5) (6)	e' = (II.)	$\frac{R - Z}{2\,L}$	$\frac{R + Z}{2\,L}$	1
10-o (5)	Same as 10-n before gap sparkover Same as 10-c after sparkover			

*See Bibliography references.

TABLE II

Fig.	Equation	α	β	A
11-a	$e' = \frac{Z_2 - Z_1}{Z_2 + Z_1} e$			
(1)	$e'' = \frac{2 Z_2}{Z_2 + Z_1} e$			
11-b	$e' = \frac{1 - Z_1 Y_0}{1 + Z_1 Y_0} e$	Y_0 = total admittance of all outgoing lines in parallel		
(1)	$e'' = \frac{2}{1 + Z_1 Y_0} e$			
11-c	$e' = \frac{Z_2 - Z_1 + R}{Z_2 + Z_1 + R} e$			
(1)	$e'' = \frac{2 Z_2}{Z_2 + Z_1 + R} e$			
11-d	$e' =$ (I.)	$\frac{Z_1 - Z_2}{L}$	$\frac{Z_1 + Z_2}{L}$	1
(1)	$e'' = \frac{\alpha - \beta}{a - \beta} (\epsilon^{-at} - \epsilon^{-\beta t})$	"	"	
11-e	$e' =$ (I.)	$\frac{1}{C (Z_1 - Z_2)}$	$\frac{1}{C (Z_1 + Z_2)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
	$e'' =$ (I.)	0	$\frac{1}{C (Z_1 + Z_2)}$	$\frac{2 Z_2}{Z_2 + Z_1}$
11-f	$e' =$ (I.)	$\frac{R (Z_1 - Z_2)}{L (R - Z_1 + Z_2)}$	$\frac{R (Z_1 + Z_2)}{L (R + Z_1 + Z_2)}$	$\frac{R + Z_2 - Z_1}{R + Z_2 + Z_1}$
	$e'' =$ (I.)	$-\frac{R Z_2}{L (R + Z_2)}$	$\frac{R (Z_1 + Z_2)}{L (R + Z_1 + Z_2)}$	$\frac{2 (Z_2 + R)}{R + Z_2 + Z_1}$
11-g	$e' =$ (II.)	$\frac{1}{2 C (Z_2 - Z_1)}$	$\frac{1}{2 C (Z_2 - Z_1)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
	$e'' =$ (II.)	0	$\frac{1}{2 C (Z_2 + Z_1)}$	$\frac{2 Z_2}{Z_2 + Z_1}$
11-h	$e' =$ (I.)	$\frac{R - Z_1 + Z_2}{R C (Z_1 - Z_2)}$	$\frac{R + Z_1 + Z_2}{R C (Z_1 + Z_2)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
	$e'' =$ (I.)	$-\frac{1}{R C}$	$\frac{R + Z_1 + Z_2}{R C (Z_1 + Z_2)}$	$\frac{2 Z_2}{Z_2 + Z_1}$
11-i	$e' =$ (II.)	$\frac{Z_2 - Z_1 + R}{2 R C (Z_2 - Z_1)}$	$\frac{Z_2 + Z_1 + R}{2 R C (Z_2 + Z_1)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
(1)	$e'' =$ (II.)	$\frac{1}{2 R C}$	$\frac{Z_2 + Z_1 + R}{2 R C (Z_2 + Z_1)}$	$\frac{2 Z_2}{Z_2 + Z_1}$
11-j	$e' =$ (I.)	$\frac{Z_1 - Z_2 - R}{L}$	$\frac{Z_1 + Z_2 + R}{L}$	1
	$e'' = \frac{-2 Z_2}{L (a - \beta)} (\epsilon^{-at} - \epsilon^{-\beta t})$		$\frac{Z_1 + Z_2 + R}{L}$	

Traveling Waves on Multi-Conductor Circuits

Equations (**9**) and (**10**) of Appendix II are the two general equations which must be satisfied by the incident, reflected, and transmitted waves on all wires at the transition points of the general system represented in Fig. 6. Therefore the solution of these simultaneous equations yields the reflected and transmitted waves in terms of the incident waves and the circuit constants of the lines and the transition points. These equations are considerably simplified when the networks N_{12}, N_{23}, etc., connecting the different lines at the junctions are equal to zero. In that case they become

$$(1 + N_r U_r) [Y_{r1} (e_1 - e_1') + \ldots \ldots + Y_{rn} (e_n - e_n')] - N_r (e_r + e_r') = (y_{r1} e_1'' + \ldots \ldots + y_{rn} e_n'')$$

$$(e_r + e_r') - U_r [Y_{r1} (e_1 - e_1') + \ldots \ldots + Y_{rn} (e_n - e_n')] = e_r'' + W_r (y_{r1} e_1'' + \ldots \ldots + Y_{rn} e_n'')$$

where n is the total number of conductors of the system, and r is any particular conductor. Each of the above equations must be written for each r so as to provide the necessary $2\,n$ simultaneous equations from which the $2\,n$ unknowns $(e_1' \ldots \ldots e_n', e_1'' \ldots \ldots e_n'')$ can be found. A few examples will be given to illustrate the use of these equations. In the interests of brevity, the illustrations are for a two-wire and ground circuit.

Fig. 13a. One of two lines suddenly terminates.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$y_{11} = 1/z_{11},\; y_{22} = y_{12} = 0$$

Substituting these values in the general equation there is

$$\left.\begin{aligned} Y_{11} (e_1 - e_1') + Y_{12} (e_2 - e_2') &= y_{11} e_1'' \\ Y_{21} (e_1 - e_1') + Y_{22} (e_2 - e_2') &= 0 \\ (e_1 + e_1') &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right\}$$

The solution of these simultaneous equations gives

$$e_1' = \frac{z_{11} - Z_{11}}{z_{11} + Z_{11}}\, e_1$$

$$e_2' = e_2 - \frac{2\,Z_{12}}{z_{11} + Z_{11}}\, e_1$$

$$e_1'' = \frac{2\,z_{11}}{z_{11} + Z_{11}}\, e_1$$

where

$$Z_{11} = \frac{Y_{22}}{Y_{11}\,Y_{22} - Y_{12}^2}$$

TABLE III

Fig.	Equation	α	β	A
12-a	$e' = \dfrac{Z_2 R - Z_1 R - Z_1 Z_2}{Z_2 R + Z_1 R + Z_1 Z_2}$ $e'' = \dfrac{2\,Z_2\,R}{Z_2 R + Z_1 R + Z_1 Z_2}$			
12-b (1)	$e' = $ (I.) $e'' = \dfrac{\alpha + \beta}{a - \beta}\,\dfrac{E}{\alpha}\,(a\,\epsilon^{-at} - \beta\,\epsilon^{-\beta t})$	$\dfrac{Z_1 Z_2}{L\,(Z_2 - Z_1)}$	$\dfrac{Z_1 Z_2}{L\,(Z_2 + Z_1)}$	$\dfrac{Z_2 - Z_1}{Z_2 + Z_1}$
12-c (1)	$e' = $ (I.) $e'' = E\,\dfrac{\alpha + \beta}{a - \beta}\,(\epsilon^{-at} - \epsilon^{-\beta t})$	$\dfrac{Z_2 - Z_1}{Z_1 Z_2 C}$ "	$\dfrac{Z_2 + Z_1}{Z_1 Z_2 C}$ "	-1
12-d	$e' = $ (I.) $e'' = e + e'$	$\dfrac{R\,Z_1 Z_2}{L\,(R Z_2 - R Z_1 - Z_1 Z_2)}$	$\dfrac{R\,Z_1 Z_2}{L\,(R Z_2 + R Z_1 + Z_1 Z_2)}$	β/α
12-e	$e' = $ (I.) $e'' = -E\,\dfrac{\alpha + \beta}{a - \beta}\,(\epsilon^{-at} - \epsilon^{-\beta t})$	$\dfrac{R Z_2 - R Z_1 - Z_1 Z_2}{Z_1 Z_2 R C}$	$\dfrac{R Z_2 + R Z_1 + Z_1 Z_2}{Z_1 Z_2 R C}$	-1
12-f	$e' = $ (II.) $e'' = e + e'$	$\dfrac{Z_1 - Z_2}{2\,Z_1 Z_2 C}$	$\dfrac{Z_1 + Z_2}{2\,Z_1 Z_2 C}$	-1
12-i (1)	$e' = $ (II.) $e'' = e + e'$	$\dfrac{Z_1 R - Z_2 R + Z_1 Z_2}{2\,Z_1 Z_2 R C}$	$\dfrac{Z_1 R + Z_2 R + Z_1 Z_2}{2\,Z_1 Z_2 R C}$	-1
12-g (2)	$e' = $ (II.) $e'' = $ (II.)	See reference	See reference	See reference
12-h (2)	$e' = $ (II.) $e'' = $ (II.)	See reference	See reference	See reference
12-j (2)	$e' = $ (II.) $e'' = $ (II.)	See reference	See reference	See reference

$$Z_{12} = \frac{-Y_{12}}{Y_{11}Y_{22} - Y_{12}^2}$$

If e_2 was induced on line No. 2 by e_1 on line No. 1 then

$$e_2 = \frac{Z_{12}}{Z_{11}} e_1$$

Or conversely, if e_1 was induced by e_2, then

$$e_1 = \frac{Z_{12}}{Z_{22}} e_2$$

If, as would likely be the case, the line to the right is simply a continuation of No. 1 wire, then

$$z_{11} = Z_{11}$$

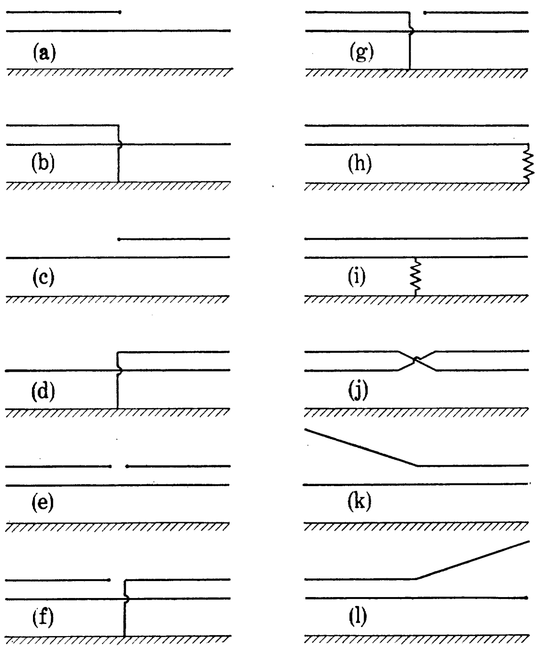

Fig. 13—Transition Points of a Double Circuit

and the equations become

$$e_1' = 0$$

$$e_2' = e_2 - \frac{Z_{12}}{Z_{11}} e_1$$

$$e_1'' = e_1$$

that is, there is no reflection on line No. 1, and the full wave is transmitted. In this case, had e_2 been induced by e_1 there would be no reflection on No. 2 conductor either.

Fig. 13b. One of two lines is terminated and grounded.

$$N_1 = U_1 = U_2 = W_1 = W_2 = 0, \quad N_2 = \infty$$

$$y_{11} = 1/z_{11}, \quad y_{22} = y_{12} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' \\ (e_2 + e_2') &= 0 \\ (e_1 + e_1') &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Solving these simultaneous equations there is

$$e_1' = \frac{Y_{11} - y_{11}}{Y_{11} + y_{11}} e_1 + \frac{2 Y_{12}}{Y_{11} + y_{11}} e_2$$

$$e_2' = -e_2$$

$$e_1'' = \frac{2 Y_{11} e_1}{Y_{11} + y_{11}} + \frac{2 Y_{12}}{Y_{11} + y_{11}} e_2$$

Fig. 13c. Isolated conductor introduced.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$Y_{11} = 1/Z_{11}, \quad Y_{12} = Y_{22} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') &= y_{11} e_1'' + y_{12} e_2'' \\ 0 &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{z_{11} - Z_{11}}{z_{11} + Z_{11}} e_1$$

$$e_1'' = \frac{2 z_{11}}{z_{11} + Z_{11}} e_1$$

$$e_2'' = \frac{z_{12}}{z_{11}} e_1'' = \frac{2 z_{12}}{z_{11} + Z_{11}} e_1$$

Thus if No. 1 is a through conductor, so that $z_{11} = Z_{11}$, there is no reflection.

Fig. 13d. Grounded conductor introduced.

$$N_1 = U_1 = U_2 = W_1 = W_2 = 0, \quad N_2 = \infty$$

$$Y_{11} = 1/Z_{11}, \; Y_{12} = Y_{22} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') &= y_{11} e_1'' + y_{12} e_2'' \\ e_2 + e_2' &= 0 \\ (e_1 + e_1') &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{Y_{11} - y_{11}}{Y_{11} + y_{11}} e_1$$

$$e_1'' = \frac{2 Y_{11}}{Y_{11} + y_{11}} e_1$$

$$e_2'' = 0$$

Fig. 13e. Break in one conductor.

$$N_1 = N_2 = U_1 = U_2 = W_1 = 0, \quad W_2 = \infty$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ 0 &= y_{21} e_1'' + y_{22} e_2'' \end{aligned}\right\}$$

Therefore, taking

$$Y_{11} = y_{11}, \; Y_{22} = y_{22}, \text{ and } Y_{12} = Y_{21} = y_{12} = y_{21}$$

$$e_1' = 0$$

$$e_2' = e_2 - \frac{z_{12}}{z_{11}} e_1$$

$$e_1'' = e_1$$

$$e_2'' = \frac{z_{12}}{z_{11}} e_1$$

Fig. 13f. Broken line—far section grounded.

$$U_1 = W_1 = W_2 = N_1 = 0, \quad U_2 = N_2 = \infty$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= 0 \\ (e_1 + e_1') &= e_1'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= 0 \end{aligned}\right\}$$

Therefore, since $e_2'' = 0$

$$e_1' = \frac{1 - Z_{11} y_{11}}{1 + Z_{11} y_{11}} e_1$$

$$e_2' = e_2 + \frac{2 Z_{12} y_{11}}{1 + Z_{11} y_{11}} e_1$$

$$e_1'' = \frac{2}{1 + Z_{11} y_{11}} e_1$$

$$e_2'' = 0$$

Fig. 13g. Broken line—near section grounded.

$$U_1 = U_2 = W_1 = N_1 = 0, \quad W_2 = N_2 = \infty$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ e_2 + e_2' &= 0 \\ e_1 + e_1' &= e_1'' \\ 0 &= y_{21} e_1'' + y_{22} e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{z_{11} Y_{11} - 1}{z_{11} Y_{11} + 1} e_1 + \frac{2 Y_{12} z_{11}}{z_{11} Y_{11} + 1} e_2$$

$$e_2' = - e_2$$

$$e_1'' = \frac{2 z_{11}}{z_{11} Y_{11} + 1} (Y_{11} e_1 + Y_{12} e_2)$$

$$e_2'' = \frac{z_{12}}{z_{11}} e_1'' = \frac{2 z_{12}}{z_{11} Y_{11} + 1} (Y_{11} e_1 + Y_{12} e_2)$$

Fig. 13h. One line grounded through a resistor at end of line.

$$U_1 = U_2 = N_2 = 0, \quad N_1 = 1/R, \quad y_{11} = y_{22} = y_{22} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') - (e_1 + e_1')/R &= 0 \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= 0 \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{R - Z_{11}}{R + Z_{11}} e_1$$

$$e_2' = e_2 - \frac{2 Z_{12}}{R + Z_{11}} e_1$$

If $R = Z_{11}$, then $e_1' = 0$ and there is no reflected wave on No. 1 wire. However, a wave

$$e_2' = e_2 - \frac{Z_{12}}{Z_{11}} e_1$$

is reflected on No. 2 wire.

Fig. 13i. Resistance ground on one wire.

$$U_1 = U_2 = W_1 = W_2 = N_2 = 0, \quad N_1 = 1/R$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' + (e_1 + e_1')/R \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ e_1 + e_1' &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{- Z_{11}}{2 R + Z_{11}} e_1$$

$$e_2' = \frac{- Z_{12}}{2 R + Z_{11}} e_1$$

$$e_1'' = \frac{2 R}{2 R + Z_{11}} e_1$$

$$e_2'' = e_2 - \frac{Z_{12}}{2 R + Z_{11}} e_1$$

These equations are of importance in connection with the theory of ground wires.[3]

Fig. 13j. Transposition of a line.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$y_{11} = Y_{22}, \; y_{22} = Y_{11}, \; y_{12} = y_{21} = Y_{12} = Y_{21}$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ e_1 + e_1' &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{(Y_{11} - Y_{22})}{(Y_{11} + Y_{22})^2 - 4 Y_{12}^2} [(Y_{11} + Y_{22}) e_1 + 2 Y_{12} e_2]$$

$$e_2' = \frac{-(Y_{11} - Y_{22})}{(Y_{11} + Y_{22})^2 - 4 Y_{12}^2} [2 Y_{12} e_1 + (Y_{11} + Y_{22}) e_2]$$

$$e_1'' = e_1' + e_1$$

$$e_2'' = e_2' + e_2$$

If the two conductors are in the same horizontal plane so that $Y_{11} = Y_{22}$, then there are no reflections.

If the two incident waves are alike, that is $e_1 = e_2 = e$ then

$$e_1' = - e_2' = \frac{(Y_{11} - Y_{22}) e}{Y_{11} + Y_{22} - 2 Y_{12}}$$

Fig. 13k. Line entering a section parallel to another line.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$Y_{12} = 0, \quad Z_{11} = z_{11}, \quad Z_{22} = z_{22}$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{[(Y_{11} - y_{11})(Y_{22} + y_{22}) + y_{12}^2] e_1 + 2 y_{12} Y_{22} e_2}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - y_{12}^2}$$

$$e_2' = \frac{[(Y_{11} + y_{11})(Y_{22} - y_{22}) + y_{12}^2] e_2 - 2 y_{12} Y_{11} e_1}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - y_{12}^2}$$

$$e_1'' = \frac{2\,Y_{11}\,(Y_{22} + y_{22})\,e_1 - 2\,y_{12}\,Y_{22}\,e_2}{(Y_{11} + y_{11})\,(Y_{22} + y_{22}) - y_{12}^2}$$

$$e_2'' = \frac{2\,Y_{22}\,(Y_{11} + y_{11})\,e_2 - 2\,y_{12}\,Y_{11}\,e_1}{(Y_{11} + y_{11})\,(Y_{22} + y_{22}) - y_{12}^2}$$

In a case of this kind it is highly improbable that both e_1 and e_2 would exist simultaneously, so that the equation could be simplified to that extent.

Fig. 13l. Line leaving a section parallel to another line.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$y_{12} = 0, \quad Z_{11} = z_{11}, \quad Z_{22} = z_{22}$$

$$\left.\begin{aligned} Y_{11}\,(e_1 - e_1') + Y_{12}\,(e_2 - e_2') &= y_{11}\,e_1'' \\ Y_{21}\,(e_1 - e_1') + Y_{22}\,(e_2 - e_2') &= y_{22}\,e_2'' \\ e_1 + e_1' &= e_1'' \\ e_2 + e_2' &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{[(Y_{11} - y_{11})\,(Y_{22} + y_{22}) - Y_{12}^2]\,e_1 + 2\,Y_{12}\,y_{22}\,e_2}{(Y_{11} + y_{11})\,(Y_{22} + y_{22}) - Y_{12}^2}$$

$$e_2' = \frac{[(Y_{11} + y_{11})\,(Y_{22} - y_{22}) - Y_{12}^2]\,e_2 + 2\,Y_{12}\,y_{11}\,e_1}{(Y_{11} + y_{11})\,(Y_{22} + y_{22}) - Y_{12}^2}$$

$$e_1'' = \frac{2\,[Y_{11}\,(Y_{22} + y_{22}) - Y_{12}^2]\,e_1 + 2\,Y_{12}\,y_{22}\,e_2}{(Y_{11} + y_{11})\,(Y_{22} + y_{22}) - Y_{12}^2}$$

$$e_2'' = \frac{2\,[Y_{22}\,(Y_{11} + y_{11}) - Y_{12}^2]\,e_2 + 2\,Y_{12}\,y_{11}\,e_1}{(Y_{11} + y_{11})\,(Y_{22} + y_{22}) - Y_{12}^2}$$

Successive Reflections

There are many important problems, such as in the theory of ground wires,[3] the effect of short lengths of cable,[11] trunk lines tapped at short intervals, etc.,[4] where it is necessary to consider a number of successive reflections of traveling waves. Sometimes it is exceedingly difficult to keep track of the multiplicity of these reflections. A lattice has therefore been devised which shows at a glance the position and direction of motion of every incident, reflected and refracted wave on the system at every instant of time. In addition, this lattice provides the means for calculating the shape of all reflected and transmitted waves and gives a complete history of their past experience. Even the effects of attenuation and wave distortion can be entered on the lattice, if the defining functions are known.

The principle of the reflection lattice is illustrated in Fig. 14. Three junctions, Nos. 1, 2, and 3 placed at uneven intervals along the line are shown. These junctions may consist of any combinations of impedances in series with the line or shunted to ground. The circuits between junctions may be either overhead lines or cables; having in general, different surge impedances, velocities of wave propagation, and attenuation factors. To construct the lattice, lay off the junctions to scale at intervals equal to the times of passage of the wave on each section between junctions. Now choose a suitable vertical time scale, shown in Fig. 14 at the left of the lattice, and draw in the diagonals. At the top of the lattice, at any convenient place centered on the junctions, place indicators with the reflection and refraction operators marked on them. In the notation of this paper these indicators are shown as little double-headed arrows marked as follows:

- a = reflection operator for waves approaching from the left.
- a' = reflection operator for waves approaching from the right.
- b = refraction operator for waves approaching from the left.
- b' = refraction operator for waves approaching from the right.
- β, α = attenuation factors for section between junctions

It is understood, of course, that these operators are

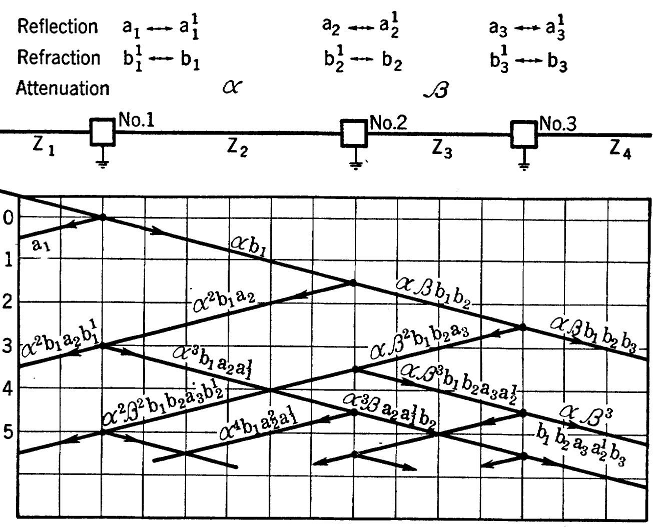

Fig. 14—Lattice for Computing Successive Reflections

operational expressions involving the impedance functions of the junctions, and no restrictions are placed on their generality. Now, starting at the origin of the initial incident wave at the top of the lattice, obtain the reflected and refracted wave at each junction by applying the operators of that junction to the incident wave arriving there, and proceed until the lattice has been completed. It will be observed that:

1. All waves travel downhill.
2. The position of any wave at any time is determined from the vertical time scale at the left of the lattice.
3. The total potential at any point at any instant of time is the superposition of all the waves which have arrived at that point up until that instant of time, displaced in position from each other by intervals equal to the instants of their time of arrival.
4. The previous history of any wave is easily traced,

that is, where it came from and just what other waves went into its composition.

5. Attenuation is included.

6. If it is desired to carry the computations to a point where it is not practical to place the various operators directly on the lattice itself, then the arms may be numbered and the operational expressions tabulated in a suitable table. No difficulty is encountered in this practise, and sometimes it is possible to devise purely tabular methods which can be filled in automatically.

Appendix I

GENERAL EQUATIONS OF A MULTI-CONDUCTOR SYSTEM

The conventional treatment of transmission line transients is based on consideration of a single wire and its return, and ignores the presence of other conductors. However, there are many cases in the study of traveling waves where the effect of the other conductors cannot be neglected. Sometimes their influence is so vital as to completely change the characteristics of the phenomenon, and entirely erroneous results are obtained if they are not considered. Problems of this type are of special engineering interest in connection with the design of ground wires and other protective schemes, and in general, in the study of mutual effects due to traveling waves. This appendix is a generalization and extention of the method of attack given in a previous paper,[3] in that line-to-line, line-to-ground, and series resistances are included in the analysis.

The General Differential Equation. Fig. 6 shows a system of n transmission line conductors, parallel to each other and to the ground plane, and mutually coupled electromagnetically, so that the effects of currents and potentials on any wire are felt on all the other wires. The circuit constants involved are shown in Fig. 7. Associated with each unit length of line and conductors r and s there is

L_{rr} = self inductance coefficient of conductor r
L_{rs} = mutual inductance coefficient between r and s
K_{rr} = self capacitance coefficient of conductor r
K_{rs} = mutual capacitance coefficient between r and s
R_r = series resistance of conductor r
g_{rr} = leakage conductance to ground of conductor r
g_{rs} = leakage conductance between r and s

It will also be convenient to introduce the notation

$$\begin{aligned} G_{rr} &= g_{r1} + g_{r2} + g_{r3} + \ldots\ldots + g_{rn} \\ G_{rs} &= G_{sr} = -g_{rs} = -g_{sr} \\ Z_{rr} &= (R_r + p\,L_{rr}) \\ Z_{rs} &= p\,L_{rs} \\ Y_{rr} &= (G_{rr} + p\,K_{rr}) \\ Y_{rs} &= (G_{rs} + p\,K_{rs}) \\ p &= \partial/\partial t \end{aligned}$$

The charges per unit length on the conductors in terms of Maxwell's electrostatic coefficients and the potentials $e_1, e_2, \ldots, e_n$ are

$$\left.\begin{aligned} Q_1 &= K_{11}\,e_1 + K_{12}\,e_2 + \ldots\ldots + K_{1n}\,e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_n &= K_{n1}\,e_1 + K_{n2}\,e_2 + \ldots\ldots + K_{nn}\,e_n \end{aligned}\right\} \quad (1)$$

The magnetic fluxes linking the conductors per unit length in terms of the inductance coefficients and the currents $i_1, i_2, \ldots, i_n$ are

$$\left.\begin{aligned} \phi_1 &= L_{11}\,i_1 + L_{12}\,i_2 + \ldots\ldots + L_{1n}\,i_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_n &= L_{n1}\,i_1 + L_{n2}\,i_2 + \ldots\ldots + L_{nn}\,i_n \end{aligned}\right\} \quad (2)$$

The leakage currents flowing to ground and to the other conductors are

$$\left.\begin{aligned} i_1' &= g_{11}\,e_1 + g_{12}\,(e_1 - e_2) + \ldots + g_{1n}\,(e_1 - e_n) \\ &= G_{11}\,e_1 + G_{12}\,e_2 + \ldots + G_{1n}\,e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ i_n' &= g_{n1}\,(e_n - e_1) + g_{n2}\,(e_n - e_2) + \ldots + g_{nn}\,e_n \\ &= G_{n1}\,e_1 + G_{n2}\,e_2 + \ldots + G_{nn}\,e_n \end{aligned}\right\} \quad (3)$$

The differential equations of the first conductor are

$$-\frac{\partial e_1}{\partial x} = \frac{\partial \phi_1}{\partial t} + R_1\,i_1 = Z_{11}\,i_1 + Z_{12}\,i_2 + \ldots\ldots + Z_{1n}\,i_n \quad (4)$$

$$-\frac{\partial i_1}{\partial x} = \frac{\partial Q_1}{\partial t} + i_1' = Y_{11}\,e_1 + Y_{12}\,e_2 + \ldots\ldots + Y_{1n}\,e_n \quad (5)$$

Differentiating equation (4) with respect to x and substituting the equations of type (5) there is

$$\begin{aligned} \frac{\partial^2 e_1}{\partial x^2} &= Z_{11}\,(Y_{11}\,e_1 + Y_{12}\,e_2 + \ldots + Y_{1n}\,e_n) \\ &+ Z_{12}\,(Y_{21}\,e_1 + Y_{22}\,e_2 + \ldots + Y_{2n}\,e_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ Z_{1n}\,(Y_{n1}\,e_1 + Y_{n2}\,e_2 + \ldots + Y_{nn}\,e_n) \\ &= (Z_{11}\,Y_{11} + Z_{12}\,Y_{21} + \ldots + Z_{1n}\,Y_{n1})\,e_1 \\ &+ (Z_{11}\,Y_{12} + Z_{12}\,Y_{22} + \ldots + Z_{1n}\,Y_{n2})\,e_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ (Z_{11}\,Y_{n1} + Z_{12}\,Y_{2n} + \ldots + Z_{1n}\,Y_{nn})\,e_n \\ &= J_{11}\,e_1 + J_{12}\,e_2 + \ldots + J_{1n}\,e_n \end{aligned} \quad (6)$$

where

$$\begin{aligned} J_{rs} &= Z_{r1}\,Y_{1s} + Z_{r2}\,Y_{2s} + Z_{r3}\,Y_{3s} + \ldots\ldots + Z_{rn}\,Y_{ns} \\ &= Z_{1r}\,Y_{1s} + Z_{2r}\,Y_{2s} + Z_{3r}\,Y_{3s} + \ldots\ldots + Z_{nr}\,Y_{ns} \end{aligned} \quad (7)$$

Let

$$A_{rr} = \left(J_{rr} - \frac{\partial^2}{\partial x^2} \right) \quad (8)$$

Then the complete set of differential equations for the n conductors may be written in the symbolic form

$$\left.\begin{aligned} 0 &= A_{11}\,e_1 + J_{12}\,e_2 + \ldots\ldots + J_{1n}\,e_n \\ 0 &= J_{21}\,e_1 + A_{22}\,e_2 + \ldots\ldots + J_{2n}\,e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 &= J_{n1}\,e_1 + J_{n2}\,e_2 + \ldots\ldots + A_{nn}\,e_n \end{aligned}\right\} \quad (9)$$

where the J's are operators in the time derivative $p = d/dt$, and the A's are operators in both the time and space derivatives. Solving these n simultaneous equations for any e, there results a determinate of which the numerator is zero on account of having a column of zeros. In order, therefore, that a solution other than zero can exist, it is necessary that the denominator also be zero (on the assumption that the indeterminate so formed will evaluate to a finite value). Therefore, there must be

$$\begin{vmatrix} A_{11} J_{12} \ldots\ldots J_{1n} \\ J_{21} A_{22} \ldots\ldots J_{2n} \\ \ldots\ldots\ldots \\ J_{n1} J_{n2} \ldots\ldots A_{nn} \end{vmatrix} e = 0 \qquad (10)$$

Now, dropping subscripts

$$Z\,Y = (R + p\,L)(G + p\,K) = L\,K\,p^2 + (L\,G + R\,K)p + R\,G \qquad (11)$$

so that ultimately, the expansion of (**10**) will lead to a polynominal of degree n in $(\partial^2/\partial\,x^2)$ and degree n in $p^2 = \partial^2/\partial\,t^2$. The solution of this partial differential equation is the most general solution for a system of parallel conductors.

There are three conditions under which equation (**10**) may be considerably simplified in obtaining a solution. These are I, *The No-Loss Line,* II, *The Completely Transposed Line,* and III, *The Alternating Current Solution.*

Case I. The No-Loss Line. The solution for this case has been given in a recent paper,[3] and will not be repeated here. However, the following discussion of the energy relationships was not included in the previous paper.

Consider a system of n potential and current waves $(e_1 \ldots . e_n,\ i_1 \ldots . i_n)$. From electrostatics (see "Electricity and Magnetism" by J. H. Jeans, page 94) and equations (**1**) of Appendix I of this paper, the total energy residing in the electrostatic field is

$$W_e = \frac{1}{2} \int (Q_1\,e_1 + Q_2\,e_2 + \ldots\ldots + Q_n\,e_n)\,d\,x$$

$$= \frac{1}{2} \int (K_{11}\,e_1^2 + K_{12}\,e_2\,e_1 + \ldots\ldots + K_{1n}\,e_n\,e_1)\,d\,x$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{1}{2} \int (K_{n1}\,e_1\,e_n + K_{n2}\,e_2\,e_n + \ldots\ldots + K_{nn}\,e_n^2)\,d\,x \qquad (12)$$

where the integration is to extend over the lengths of the waves.

The electromagnetic energy (Jeans page 443) by equations (**2**) of Appendix I is

$$W_r = \frac{1}{2} \int (\phi_1\,i_1 + \phi_2\,i_2 + \ldots\ldots + \phi_n\,i_n)\,d\,x$$

$$= \frac{1}{2} \int (L_{11}\,i_1^2 + L_{12}\,i_2\,i_1 + \ldots\ldots + L_{1n}\,i_n\,i_1)\,d\,x$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{1}{2} \int (L_{n1}\,i_1\,i_n + L_{n2}\,i_2\,i_n + \ldots\ldots + L_{nn}\,i_n^2)\,d\,x \qquad (13)$$

The total energy of the waves is (considering only the waves moving in one direction, and calling the current waves i)

$$\begin{aligned} W &= \int (e_1\,i_1 + e_2\,i_2 + \ldots\ldots + e_n\,i_n)\,dt \\ &= \int (z_{11}\,i_1^2 + z_{12}\,i_2\,i_1 + \ldots\ldots + z_{1n}\,i_n\,i_1)\,dt \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \int (z_{n1}\,i_1\,i_n + z_{n2}\,i_2\,i_n + \ldots\ldots + z_{nn}\,i_n^2)\,dt \\ &= \int (y_{11}\,e_1^2 + y_{12}\,e_2\,e_1 + \ldots\ldots + y_{1n}\,e_n\,e_1)\,dt \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \int (y_{n1}\,e_1\,e_n + y_{n2}\,e_2\,e_n + \ldots\ldots + y_{nn}\,e_n^2)\,dt \end{aligned} \qquad (14)$$

But for free traveling waves on overhead lines,[3]

$$z = c\,L \qquad (15)$$

and

$$d\,x = v\,d\,t = c\,dt \qquad (16)$$

Therefore, equation (**14**) may be written

$$\begin{aligned} W &= \int (L_{11}\,i_1^2 + L_{12}\,i_2\,i_1 + \ldots\ldots + L_{1n}\,i_n\,i_1)\,d\,x \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \int (L_{n1}\,i_1\,i_n + L_{n2}\,i_2\,i_1 + \ldots\ldots + L_{nn}\,i_n^2)\,d\,x \\ &= 2\,W_i \end{aligned} \qquad (17)$$

thus proving that the energy of the system is divided equally between the electrostatic and magnetic fields for free waves traveling in one direction. While waves moving in opposite directions are passing through each other the total energy is not equally divided, but may be distributed in any proportion between the two fields. This is also true at a transition point, where the incident waves give rise to reflected waves. In such cases the energies must be computed from equations (**12**) and (**13**) and added to find the total. Equation (**14**) applies only to waves moving in the same direction, and while it serves to determine the total energy by computing the energies in each system of oppositely moving waves and adding them, it does not hold for resultant potentials and currents.

Case II. The Completely Transposed Line. If all n conductors are completely transposed with respect to each other and to the ground, and if the conductors have the same resistance, then in effect

$$L_{rr} = L\,, \quad K_{rr} = K\,, \quad G_{rr} = G$$

$$L_{rs} = L', \quad K_{rs} = K', \quad G_{rs} = G'$$

$$Z_{rr}\,Y_{rr} = (R + p\,L)\,(G + p\,K) = Z\,Y$$

$$Z_{rr}\,Y_{rs} = (R + p\,L)\,(G' + p\,K') = Z\,Y'$$

$$Z_{rs}\,Y_{rr} = p\,L'\,(G + p\,K) = Z'\,Y$$

$$Z_{rs}\,Y_{rs} = p\,L'\,(G' + p\,K') = Z'\,Y'$$

$$\begin{aligned} J_{rs} &= Z_{1r}\,Y_{1s} + Z_{2r}\,Y_{2s} + \ldots\ldots + Z_{rr}\,Y_{rs} \\ &+ \ldots\ldots + Z_{sr}\,Y_{ss} + \ldots\ldots + Z_{nr}\,Y_{ns} \\ &= Z\,Y' + Z'\,Y + (n-2)\,Z'\,Y' = J \end{aligned} \qquad (18)$$

$$\begin{aligned} J_{rr} &= Z_{1r}\,Y_{1r} + Z_{2r}\,Y_{2r} + \ldots\ldots + Z_{rr}\,Y_{rr} \\ &+ \ldots\ldots + Z_{nr}\,Y_{nr} \\ &= Z\,Y + (n-1)\,Z'\,Y' \end{aligned} \qquad (19)$$

$$A_{rr} = \left(J_{rr} - \frac{\partial^2}{\partial x^2}\right) = \left[Z\,Y + (n-1)\,Z'\,Y' - \frac{\partial^2}{\partial x^2}\right] = A \quad (20)$$

$$(A - J) = \left[(Z - Z')\,(Y - Y') - \frac{\partial^2}{\partial x^2}\right] \quad (21)$$

Hereby the determinate of equation (**10**) becomes, upon dividing each column through by J and calling $A/J = a$, subtracting adjacent columns from each other, dividing out $(a - 1)$, adding all rows to the last, and finally expanding the remaining determinate in terms of the minors of which the lower right hand elements are the cofactors

$$\begin{vmatrix} A & J & \ldots & J \\ J & A & \ldots & J \\ \ldots & & & \\ J & J & \ldots & A \end{vmatrix} e = (A-J)^{n-1}\,(A - J + n\,J) \quad (22)$$

This is the operational equation of the completely transposed line in terms of the operators A and J. Upon substituting the values of A and J from equations (**18**) and (**20**) it takes the form

$$\left(v_1^2\,p^2 + w_1\,p + u_1 - \frac{\partial^2}{\partial x^2}\right)^{n-1}\left(v_2^2\,p^2 + w_2\,p + u_2 - \frac{\partial^2}{\partial x^2}\right) e = 0 \quad (23)$$

If the line is free of losses, as well as completely transposed, then (**22**) becomes

$$\left(v_1^2\,p^2 - \frac{\partial^2}{\partial x^2}\right)^{n-1}\left(v_2^2\,p^2 - \frac{\partial^2}{\partial x^2}\right) e = 0 \quad (24)$$

This is satisfied by the equation of wave motion

$$e = f\,(x + v\,t) \quad (25)$$

which substituted in (**43**) gives

$$v = \pm\, v_1 \text{ and } v = \pm\, v_2 \quad (26)$$

thus showing that there are only two possible velocities of wave propagation on the completely transposed no-loss line. In the case of overhead conductors in air both of these velocities approach equality with the velocity of light, in agreement with the findings of the previous section.

Case III. Solution for Alternating Currents. Suppose that the line is operating under steady-state alternating-current conditions, so that the potentials on the n conductors at coordinate x are given by

$$\left.\begin{aligned} e_1 &= E_1 \sin\,(\omega\,t + \theta_1) = \text{imaginary part of } E_1\,\epsilon^{j(\omega t + \theta_1)} \\ &\ldots\ldots\ldots \\ e_n &= E_n \sin\,(\omega\,t + \theta_n) = \text{imaginary part of } E_n\,\epsilon^{j(\omega t + \theta_n)} \end{aligned}\right\} \quad (27)$$

where the amplitudes $(E_1 \ldots\ldots E_n)$ are functions of x. Substituting (**27**) in (**6**) there is

$$\frac{\partial^2\,\dot{E}_1}{\partial\,x^2} = \dot{J}_{11}\,\dot{E}_1 + \dot{J}_{12}\,\dot{E}_2 + \ldots\ldots + \dot{J}_{1n}\,\dot{E} \quad (28)$$

where

$$\left.\begin{aligned} \dot{E}_1 &= E_1\,\epsilon^{j\theta_1} \\ &\ldots\ldots \\ \dot{E}_n &= E_n\,\epsilon^{j\theta_n} \end{aligned}\right\} \quad (29)$$

$$\left.\begin{aligned} \dot{J}_{11} &= J_{11} \text{ when } p = j\,\omega \\ &\ldots\ldots \\ \dot{J}_{1n} &= J_{1n} \text{ when } p = j\,\omega \end{aligned}\right\} \quad (30)$$

The common factor $\epsilon^{j\omega t}$ has been canceled out on both sides of equation (**28**).

Now according to equation (**10**), the general differential equation is the same for every conductor of the n wire system, and therefore each $\dot{E}$ of (**28**) must follow the same function of x, but the integration constants are, in general, different for each $\dot{E}$. Since the differential equations are ordinary linear differential equations with constant coefficients of order $2\,n$ and homogeneous in $d^2/d\,x^2$ it follows that the solutions are of the form

$$\left.\begin{aligned} \dot{E}_1 &= \sum_1^n (\dot{C}_{1r}\,\epsilon^{\lambda_r x} + \dot{C}_{1r}'\,\epsilon^{-\lambda_r x}) \\ &\ldots\ldots\ldots \\ \dot{E}_n &= \sum_1^n (\dot{C}_{nr}\,\epsilon^{\lambda_r x} + \dot{C}_{nr}'\,\epsilon^{-\lambda_r x}) \end{aligned}\right\} \quad (31)$$

where the $\dot{C}$'s are complex integration constants and the λ's are the roots of equation (**10**). There are $2\,n^2$ of these integration constants, of which all except $2\,n$ are redundant. To prove this, substitute equations (**31**) into the original equations of type (**28**), obtaining n equations of the type

$$\begin{aligned} \sum_1^n \lambda_r^2\,(\dot{C}_{1r}\,\epsilon^{\lambda_r x} + \dot{C}_{1r}'\,\epsilon^{-\lambda_r x}) &= \sum_1^n \dot{J}_{11}\,(\dot{C}_{1r}\,\epsilon^{\lambda_r x} + \dot{C}_{1r}'\,\epsilon^{-\lambda_r x}) \\ &\ldots\ldots\ldots \\ &+ \sum_1^n \dot{J}_{1n}\,(\dot{C}_{nr}\,\epsilon^{\lambda_r x} + \dot{C}_{1r}'\,\epsilon^{-\lambda_r x}) \end{aligned} \quad (32)$$

Collecting terms

$$\sum_1^n \{[(\lambda_r^2 - \dot{J}_{11})\,\dot{C}_{1r} - \dot{J}_{12}\,\dot{C}_{2r} - \ldots\ldots - \dot{J}_{1n}\,\dot{C}_{nr}]\,\epsilon^{\lambda_r x} + [(\lambda_r^2 - \dot{J}_{11})\,\dot{C}_{1r}' - \dot{J}_{12}\,\dot{C}_{2r}' - \ldots\ldots - \dot{J}\,\dot{C}_{nr}']\,\epsilon^{-\lambda_r x}\} = 0 \quad (33)$$

It is now necessary to digress long enough to prove (what is probably self-evident to many) that each of the coefficients in (**33**) is individually equal to zero. In the general case let

$$A\,f_1\,(x) + B\,f_2\,(x) + \ldots\ldots + N\,f_n\,(x) = 0 \quad (34)$$

where the functions $f\,(x)$ are all different. Assuming that each of these admits of expansion as a power series in x, by Maclaurin's theorem, there is

$$A\,(a_1 + b_1\,x + c_1\,x^2 + \ldots\ldots) + \ldots\ldots + N\,(a_n + b_n\,x + c_n\,x^2 + \ldots\ldots) = 0 \quad (35)$$

Collecting terms

$$(a_1 A + a_2 B + \ldots + a_n N) + (b_1 A + b_2 B + \ldots + b_n N) x + \ldots = 0 \quad (36)$$

But by the method of indeterminate coefficients, each term of this power series in x must individually equal zero, so that there are the simultaneous equations

$$\left.\begin{aligned} a_1 A + a_2 B + \ldots + a_n N &= 0 \\ b_1 A + b_2 B + \ldots + b_n N &= 0 \\ c_1 A + c_2 B + \ldots + c_n N &= 0 \\ \text{etc.} \end{aligned}\right\} \quad (37)$$

Since the coefficients $(a, b, c \ldots)$ are entirely arbitrary, the solution of these equations leads to a determinate of which the denominator is finite and the numerator is zero (by virtue of a column of zeros), and therefore

$$A = B = \ldots = N = 0 \quad (38)$$

Thus in any equation of type (**34**) the individual coefficients are separately equal to zero.

Returning now to the equations of type (**33**), and considering all n of these equations, there is

$$\left.\begin{aligned} (\dot{J}_{11} - \lambda_r^2)\dot{C}_{1r} + \dot{J}_{12}\dot{C}_{2r} + \ldots + \dot{J}_{1n}\dot{C}_{nr} &= 0 \\ \dot{J}_{21}\dot{C}_{1r} + (\dot{J}_{22} - \lambda_r^2)\dot{C}_{2r} + \ldots + \dot{J}_{2n}\dot{C}_{nr} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{J}_{n1}\dot{C}_{1r} + \dot{J}_{n2}\dot{C}_{2r} + \ldots + (\dot{J}_{nn} - \lambda_r^2)\dot{C}_{nr} &= 0 \end{aligned}\right\} \quad (39)$$

and exactly the same relationships hold between the $\dot{C}'$ coefficients. Now in order that (**39**) may be satisfied by values of the $\dot{C}$'s other than zero, the denominator of the determinate must be equal to zero, that is

$$\begin{vmatrix} (\dot{J}_{11} - \lambda_r^2) & \dot{J}_{12} & \ldots & \dot{J}_{1n} \\ \dot{J}_{21} & (\dot{J}_{22} - \lambda_r^2) & \ldots & \dot{J}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \dot{J}_{n1} & \dot{J}_{n2} & \ldots & (\dot{J}_{nn} - \lambda_r^2) \end{vmatrix} = 0 \quad (40)$$

Therefore, if (**40**) holds, there are $(n - 1)$ independent relationships between the $\dot{C}$'s in equation (**39**), so that any $(n - 1)$ of them may be eliminated. But since there are n values of r, there will remain n integration constants that must be determined from the terminal conditions. Likewise, there will remain n arbitrary integration constants among the $\dot{C}'$ coefficients.

Thus, the n-wire transmission system has associated with it n propagation constants (λ_r) and $2n$ integration constants $\dot{C}_r$ and $\dot{C}_r'$.

If the line is completely transposed, then by equation (**22**) there are only two independent roots to the differential equations, and therefore only four integration constants.

If the line is a completely transposed three-wire line, there is, by equations (**22**), (**18**), and (**20**)

$$\left[(Z - Z')(Y - Y') - \frac{\partial^2}{\partial x^2}\right]^2 \left[(Z + 2Z')(Y + 2Y') - \frac{\partial^2}{\partial x^2}\right]\dot{E} = 0 \quad (41)$$

Therefore the propagation constants are

$$\left.\begin{aligned} \lambda_1^2 &= (Z - Z')(Y - Y') \\ \lambda_2^2 &= (Z + 2Z')(Y + 2Y') \end{aligned}\right\} \quad (42)$$

But (λ_2^2) does not satisfy (**40**), so that there are no $r = 2$ constants. However, (λ_1^2) does satisfy (**40**) and therefore the solution for a completely transposed three-wire line is

$$\left.\begin{aligned} \dot{E}_1 &= \dot{C}_{11}\epsilon^{\lambda_1 x} + \dot{C}_{11}'\epsilon^{-\lambda_1 x} \\ \dot{E}_2 &= \dot{C}_{21}\epsilon^{\lambda_1 x} + \dot{C}_{21}'\epsilon^{-\lambda_1 x} \\ \dot{E}_3 &= \dot{C}_{31}\epsilon^{\lambda_1 x} + \dot{C}_{31}'\epsilon^{-\lambda_1 x} \end{aligned}\right\} \quad (43)$$

If for the complex number (λ_1) there be substituted

$$\lambda_1 = \alpha + j\beta \quad (44)$$

then equations (**44**) may be expressed in any of the following familiar forms

$$\begin{aligned} \dot{E} &= A\epsilon^{\lambda x} + B\epsilon^{-\lambda x} \\ &= A\epsilon^{\alpha x}(\cos\beta x + j\sin\beta x) + B\epsilon^{-\alpha x}(\cos\beta x - j\sin\beta x) \\ &= (A + B)\cosh\lambda x + (A - B)\sinh\lambda x \\ &= (A + B)(\cosh\alpha x \,.\, \cos\beta x + j\sinh\alpha x \,.\, \sin\beta x) \\ &\quad + (A - B)(\sinh\alpha x \,.\, \cos\beta x + j\cosh\alpha x \,.\, \sin\beta x) \end{aligned} \quad (45)$$

This is the so-called vector solution. The actual potential as function of x and t is

$$\begin{aligned} e &= \text{imaginary part of } \dot{E}\,\epsilon^{j\omega t} \\ &= \text{imaginary part of } (A\epsilon^{\lambda x} + B\epsilon^{-\lambda x})\,\epsilon^{j\omega t} \end{aligned} \quad (46)$$

It is worth noticing from (**42**) that λ_1 is in terms of the so-called "constants to neutral" used in practical transmission line calculations. For if

h = geometric mean height above ground
s = geometric mean spacing between conductors
r = radius of conductors

then

$$Z = R + j\left(\frac{1}{2} + 2\log\frac{2h}{r}\right)10^{-9}$$

$$Z' = \quad + j\left(2\log\frac{2h}{s}\right)10^{-9}$$

$$(Z - Z') = R + j\left(\frac{1}{2} + 2\log\frac{s}{r}\right)10^{-9} \text{ ohms}$$

$$(Y - Y') = (G - G') + \frac{j}{\left(18\log\frac{s}{r}\right)10^{11}} \text{ mhos}$$

Appendix II

Behavior of Waves at a Transition Point

In a previous paper[3] equations were given for the reflected and transmitted waves on an n-wire transmission system similar to that shown in Fig. 6, but lacking the mutual connecting admittances, N_{12}, N_{13}, etc. In this appendix these admittances are included in the analysis, and the more general equations obtained. The application of these general equations is illustrated in the paper by a number of practical examples which have come up from time to time in the investigation of artificial lightning surges. Even the complicated circuit of Fig. 6 will not serve for every conceivable case, but it hardly seems profitable to generalize any further. If a particular transition point cannot be made a special case of Fig. 6, it will probably be as easy to solve it directly as to reduce it from anything more general. In any event, the procedure followed in setting up the equations and solving them is the same regardless of how complex the transition points may be.

Referring to Fig. 6 let

$Y_{11}, Y_{22}, \ldots, Y_{nn}$ = self surge admittances of lines on the left

Y_{12}, Y_{13}, etc. = mutual surge admittances of lines on the left

$y_{11}, y_{22}, \ldots, y_{nn}$ = self surge admittances of lines on the right

y_{12}, y_{13}, etc. = mutual surge admittances of lines on the right

$U_1, U_2, \ldots, U_n$ = series impedance network on the left

$W_1, W_2, \ldots, W_n$ = series impedance network on the right

$N_1, N_2, \ldots, N_n$ = admittances to ground

N_{12}, N_{23}, etc. = admittances from junction to junction

e, i = potential and current incident waves

e', i' = potential and current reflected waves

e'', i'' = potential and current transmitted waves

When the incident waves arrive at the transition points, they give rise to reflected and transmitted waves which satisfy the general equations of the transmission line, and are in accord with Kirchhoff's laws and the conditions of current and voltage continuity at the junctions.

The total potential at the junction of any incoming line r is the sum of the incident and reflected waves on that line

$$(e_r + e_r') \quad (1)$$

and the total current is[3]

$$(i_r + i_r') = Y_{r1}(e_1 - e_1') + \ldots + Y_{rn}(e_n - e_n') \quad (2)$$

The potential across the admittance N_r is

$$E_r = (e_r + e_r') - U_r(i_r + i_r') \quad (3)$$

and the current through N_r therefore is

$$I_r = N_r E_r \quad (4)$$

The current transmitted to the outgoing line is

$$i_r'' = y_{r1} e_1'' + y_{r2} e_2'' + \ldots + y_{rn} e_n'' \quad (5)$$

and the current transferred to the other junction is

$$I_r' = N_{1r}(E_r - E_1) + N_{2r}(E_r - E_2) + \ldots + N_{nr}(E_r - E_n) \quad (6)$$

The condition of current continuity is

$$i_r + i_r' = i_r'' + I_r + I_r' \quad (7)$$

The potential wave transmitted to the outgoing line is

$$e_r'' = E_r - W_r i_r'' \quad (8)$$

Substituting (2), (3), (4), (5) and (6) in (7) and rearranging, there is

$$[Y_{r1} + Y_{r1} U_r (N_r + N_{1r} + \ldots + N_{nr}) - Y_{11} N_{1r} U_1 - \ldots - Y_{n1} N_{nr} U_n](e_1 - e_1')$$
$$\ldots\ldots\ldots\ldots$$
$$+ [Y_{rn} + Y_{rn} U_r (N_r + N_{1r} + \ldots + N_{nr}) - Y_{1n} N_{1r} U_1 - \ldots - Y_{nn} N_{nr} U_n](e_n - e_n')$$
$$+ N_{1r}(e_1 + e_1') + \ldots + N_{nr}(e_n + e_n') - (N_r + N_{1r} + \ldots + N_{nr})(e_r + e_r')$$
$$= (y_{r1} e_1'' + \ldots + y_{rn} e_n'') \quad (9)$$

Substituting (2), (3), (4), (5) and (6) in (8) and rearranging, there is

$$(e_r + e_r') - U_r[Y_{r1}(e_1 - e_1') + \ldots + Y_{rn}(e_n - e_n')] = e_r'' + W_r(y_{r1} e_1'' + \ldots + y_{rn} e_n'') \quad (10)$$

For an n-wire system n equations of type (9) and n equations of type (10) can be written, and these $2n$ simultaneous equations suffice for the determination of the $2n$ unknowns ($e_1' \ldots e_n'$, $e_1'' \ldots e_n''$). The other quantities may then be found from equations (1) to (8). These equations are therefore sufficient to completely formulate the behavior of the incident, reflected, and transmitted waves at a general transition point. Some simplifications and examples are given below.

Mutual Connecting Networks Removed. Suppose that N_{12}, N_{23}, etc., are all zero. Then equations (9) and (10) reduce to

$$(1 + N_r U_r)[Y_{r1}(e_1 - e_1') + \ldots + Y_{rn}(e_n - e_n')] - N_r(e_r + e_r') = (y_{r1} e_1'' + \ldots + y_{rn} e_n'') \quad (11)$$

$$(e_r + e_r') - U_r[Y_{r1}(e_1 - e_1') + \ldots + Y_{rn}(e_n - e_n')] = e_r'' + W_r(y_{r1} e_1'' + \ldots + y_{rn} e_n'') \quad (12)$$

These are the general equations derived in a previous paper.[3]

Single Wire Line. In this case only e_1, e_1' and e_1'' exist, and equations (11) and (12) become

$$(1 + N_1 U_1) Y_{11}(e_1 - e_1') - N_1(e_1 + e_1') = y_{11} e_1'' \quad (13)$$

$$- U_1 Y_{11}(e_1 - e_1') + (e + e_1') = (1 + W_1 y_{11}) e_1'' \quad (14)$$

Solving these two simultaneous equations for the reflected and transmitted waves, substituting $Z_{11} = 1/Y_{11}$ and $z_{11} = 1/y_{11}$, and dropping subscripts, there is

$$e' = \frac{(z + W)(1 + NU) + U - Z - ZN(z + W)}{(z + W)(1 + NU) + U + Z + ZN(z + W)} e \quad (15)$$

$$e'' = \frac{2z}{(z + W)(1 + NU) + U + Z + ZN(z + W)} e \quad (16)$$

The conventional traveling wave theory is based on a single-wire line and is expressed by the above equations. In terms of the total impedance at the transition point,

$$Z_o = U + \frac{1}{\left(N + \frac{1}{W + z}\right)} = \frac{U + (1 + N\,U)\,(W + z)}{1 + N\,(W + z)} \quad (17)$$

the above equations take the more familiar form

$$e' = \frac{Z_o - Z}{Z_o + Z}\,e \quad (18)$$

$$e'' = \frac{1}{1 + N\,(W + z)} \quad \frac{2\,z}{Z_o + Z}\,e \quad (19)$$

Equations and curves of a great many combinations have been worked out as special cases of these expressions. Those shown in Figs. 10, 11, and 12 are in the readily accessible literature. It will be noticed that many of the combinations illustrated are special cases of the more complicated circuits, merely by substituting limiting values (zero or infinite) for the constants R, L, C, and Z as required. Moreover, each of these cases is of practical importance in the study of high-voltage surges.

Energy Relationships at the Junctions. The energy of a free traveling wave is given by equations (**12**) to (**17**) of Appendix I. During the time that the incident waves are at the junction a redistribution of energy is taking place. The division of energy during this transition period furnishes a valid check on the reflection, refraction, and transfer operators, and is of interest on its own account. At any time t, counting from the instant when the system of incident waves $(e_1 \ldots . e_n)$ arrive at the junction, there is

$$\int_t^\infty (e_1\,i_1 + \ldots . + e_n\,i_n)\,dt$$

= energy remaining in the incident waves **(21)**

$$-\int_o^t (e_1'\,i_1' + \ldots . + e_n'\,i_n')\,dt$$

= energy in the reflected waves **(22)**

$$\int_o^t (e_1''\,u_1'' + \ldots . + e_n''\,i_n'')\,dt$$

= energy in the transmitted waves **(23)**

$$\int_o^t (E_1\,I_1 + \ldots . + E_n\,I_n)\,dt$$

= energy absorbed by the networks $(N_1 \ldots . N_n)$ **(24)**

$$\int_o^t [(e_1 + e_1' - E_1)\,(i_1' + i_1) + \ldots + (e_n + e_n' - E_n)\,(i_n' + i_n)]\,dt$$

= energy absorbed by the networks $(U_1 \ldots U_n)$ **(25)**

$$\int_o^t [(E_1 - e_1'')\,i_1'' + \ldots + (E_n - e_n'')\,i_n]\,dt$$

= energy absorbed by the networks $(W_1 \ldots W_n)$ **(26)**

$$\int_o^t \sum_1^n \sum_s (E_r - E_s)\,.\,N_{rs}\,(E_r - E_s)\,dt$$

= energy absorbed in connecting networks N_{rs} **(27)**

The $\sum_s$ summation in (**27**) ranges from $s = 1$ to $s = n$, excepting $s = r$.

Equating the sum of these energies to the energy in the original incident waves, by the conservation of energy,

$$\int_o^\infty \sum_1^n e_r\,i_r\,dt = \int_t^\infty \sum_1^n e_r\,i_r\,dt - \int_o^t \sum_1^n e_r'\,i_r'\,dt$$

$$+ \int_o^t \sum_1^n e_r''\,i_r''\,dt + \int_o^t \sum_1^n E_r\,I_r\,dt$$

$$+ \int_o^t \sum_1^n (e_r + e_r' - E_r)\,(i_r + i_r')\,dt$$

$$+ \int_o^t \sum_1^n (E_r - e_r'')\,i_r''\,dt$$

$$+ \int_o^t \sum_1^n \sum_s (E_r - E_s)\,.\,N_{rs}\,(E_r - E_s)\,dt \quad (28)$$

Combining the first term on the right with the term on the left, according to the rule

$$\int_o^\infty - \int_t^\infty = \int_o^t \quad (29)$$

and discarding the integrals, there is

$$\sum_{r=1}^n [e_r\,i_r + e_r'\,i_r' - e_r''\,i_r'' - E_r\,I_r - (e_r + e_r' - E_r)(i_r + i_r')$$

$$- (E_r - e_r'')\,i_r'' - \sum_s (E_r - E_s)\,.\,N_{rs}\,(E_r - E_s)] = 0 \quad (30)$$

The currents and voltages at a transition as determined by the reflection, refraction, and transfer operators, must satisfy equation (**30**).

Bibliography

1. *Traveling Waves Due to Lightning*, L. V. Bewley, A. I. E. E. TRANS., Vol. 48, July 1929.

2. *Shunt Resistors for Reactors*, F. H. Kierstead, H. L. Rorden, and L. V. Bewley, A. I. E. E. TRANS., July 1930, p. 1161.

3. *Critique of Ground Wire Theory*, L. V. Bewley, A. I. E. E. JOURNAL, September 1930, p. 780.

4. *Attenuation and Successive Reflections of Traveling Waves*, J. C. Dowell, A. I. E. E. TRANS., Jan. 1931.

5. "Calculation of Voltage Stresses Due to Traveling Waves, with Special Reference to Choke Coils," E. W. Boehne, *General Electric Review*, Dec. 1929, Vol. 32, No. 12.

6. "Reflection of Transmission Line Surges at a Terminal Impedance," O. Brune, *General Electric Review*, May 1929, Vol. 32.

7. *Arcing Grounds and Effect of Neutral Grounding Impedance*, J. E. Clem, A. I. E. E. TRANS., July 1930, p. 970. See also discussion by L. V. Bewley.

8. "Traveling Waves," L. V. Bewley, *Journal Maryland Academy of Sciences*, Oct. 1930.

9. *Electric Oscillations in the Double Circuit Three-Phase Transmission Line*, Y. Satoh, A.I.E.E. TRANS., January 1928, p. 64.

10. "Performance of Thyrite Arresters for Any Assumed Form of Traveling Wave and Circuit Arrangement," K. B. McEachron and H. G. Brinton, *General Electric Review*, June 1930.

11. *Study of the Effect of Short Lengths of Cable on Traveling Waves*, K. B. McEachron, J. G. Hemstreet, and H. P. Seelye, A. I. E. E. TRANS., October 1930, p. 1432. Also discussions of this paper by H. G. Brinton and L. V. Bewley.

Discussion

For discussion of this paper see page 557.