TRAVELING WAVES
ON TRANSMISSION SYSTEMS

TRAVELING WAVES ON TRANSMISSION SYSTEMS

BY

L. V. BEWLEY
Power Transformer Engineering Department
General Electric Company
Pittsfield, Mass.

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PREFACE

THE theory of traveling electric waves on transmission lines and in transformer windings has undergone extensive developments in the past few years, as indicated by numerous technical papers in the *Transactions* of the American Institute of Electrical Engineers and elsewhere. In particular, the recently formed concepts of the laws of cloud discharge and consequent lightning wave formations, of the transmission and reflection of waves of arbitrary shape, and of electrical oscillations in transformer windings set up by impulse voltages, have tremendously advanced our knowledge of wave phenomena and have made possible remarkably accurate quantitative analyses of lightning and other transient phenomena. However, as in any new science, the theory of traveling waves has been built up piecemeal over a long period of time. Oliver Heaviside, C. P. Steinmetz, W. Petersen, K. W. Wagner, R. Pfiffner, R. Rudenberg, and many others have contributed much to its early development.

The consolidation of these theories in book form was undertaken to provide a convenient reference text for the use of the Advanced Course in Engineering of the General Electric Company. In carrying out the work, the attempt has been made to present a fundamental and generalized mathematical analysis of the subject, that should be of permanent reference value. Methods for evaluating the effects of any arbitrary voltage impulse are given. It is hoped, therefore, that the book will prove useful to professional engineers engaged in problems of transmission lines and machine transients, as well as to students approaching the subject for the first time.

That the reader has a working knowledge of operational calculus is presupposed, since it is so well adapted to the treatment of traveling wave theory. However, in giving a course of lectures on traveling waves, substantially as presented here, to a group of engineers unfamiliar with Heaviside's methods, I have found that sufficient operational calculus for the purpose could readily be imparted in a half dozen one-hour lectures.

The brief bibliographies included in Parts I and II, covering the development of the theory of traveling waves and transformer transients, make no pretense at completeness, but merely represent those publications with which I am familiar and have consulted in the compilation of this work. The literature is so stupendous and so scattered, and so much of it is "lost in the archives of antiquity," that I can not hope to do justice to it. But I would appreciate having my attention called to glaring omissions, so that they may be included should a subsequent edition of this book ever be called for. Dr. R. Rudenberg's excellent treatise "Elektrische Schaltvorgänge" * very well covers the development in this field up to 1926, however.

In tendering my thanks to the many friends who have offered me encouragement and valuable suggestions, it is not so much a question of where to begin as where to stop. To Mr. F. W. Peek, Jr., and to his associates, Messrs. F. F. Brand and H. O. Stephens, in the Power Transformer Department of the General Electric Company, I owe the opportunity to work along these lines, and I am greatly indebted to them for their generous extension to me of the facilities of the department in the preparation of this book. I am indebted to Messrs. P. L. Alger, J. E. Clem, and Alan Howard for reviewing the manuscript and for many helpful suggestions. Messrs. J. E. Clem and W. F. Skeats kindly gave me permission to make use of their papers in writing Chapter X, and Messrs. K. K. Palueff and J. H. Hagenguth likewise generously allowed me to incorporate material from their papers in writing Chapter XV.

In the preparation of Part II, I have drawn freely from the sources given in the Bibliography, and in addition I have enjoyed the advantage of close association with the engineers of the General Electric Company who have pioneered so much of this work. I wish to thank Dr. V. Bush and Dr. E. J. Berg for kindly permitting me to copy tables from their books to form the Appendix.

L. V. BEWLEY.

* Springer, 1926, Second Edition.

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INTRODUCTION TO PART I

THE ORIGIN, CHARACTERISTICS, AND BEHAVIOR OF TRAVELING WAVES

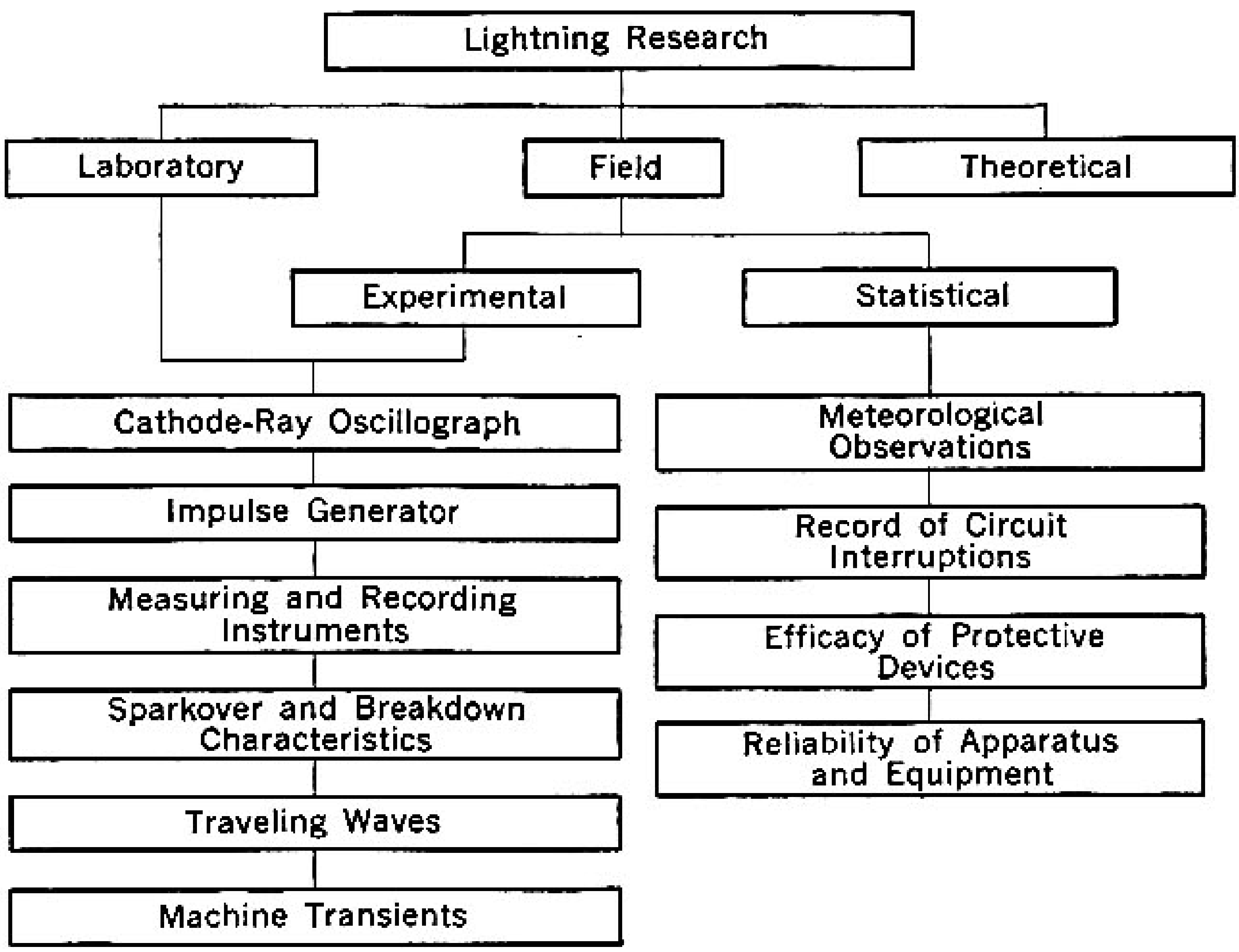
The persistent and increasing effort on the part of the electric power industry to reduce the number of outages and to preserve the best possible continuity of service has directed special attention towards the protection of transmission lines and station apparatus from the principal cause of abnormal system disturbances—lightning. During the past few years the lightning problem has been exhaustively studied from all angles, and this study has been facilitated by a commendable cooperation between the public utilities and the manufacturers of electrical apparatus, so that today the understanding of the problem is reaching a satisfactory basis, and economical means are available for practical immunity from lightning trouble.

In the following chart the principal divisions and subdivisions of lightning research are tabulated.

The rôle of the laboratory has been twofold; first, to develop suitable means for the production, control, and measurement of "artificial lightning"; and second, to study the effects of these high-voltage impulses on insulation structures and apparatus. Out of this effort have come the impulse generator, for the production of impulses of predetermined magnitude and shape; the cathode-ray oscillograph, for the photography of impulse waves; calibrated sphere, point, and rod gaps and Lichtenberg figures for the measurement of voltages; volumes of data on the sparkover and breakdown characteristics of gas, oil, and solid insulation; and finally, the experimental studies of traveling waves and high-frequency transients of apparatus.

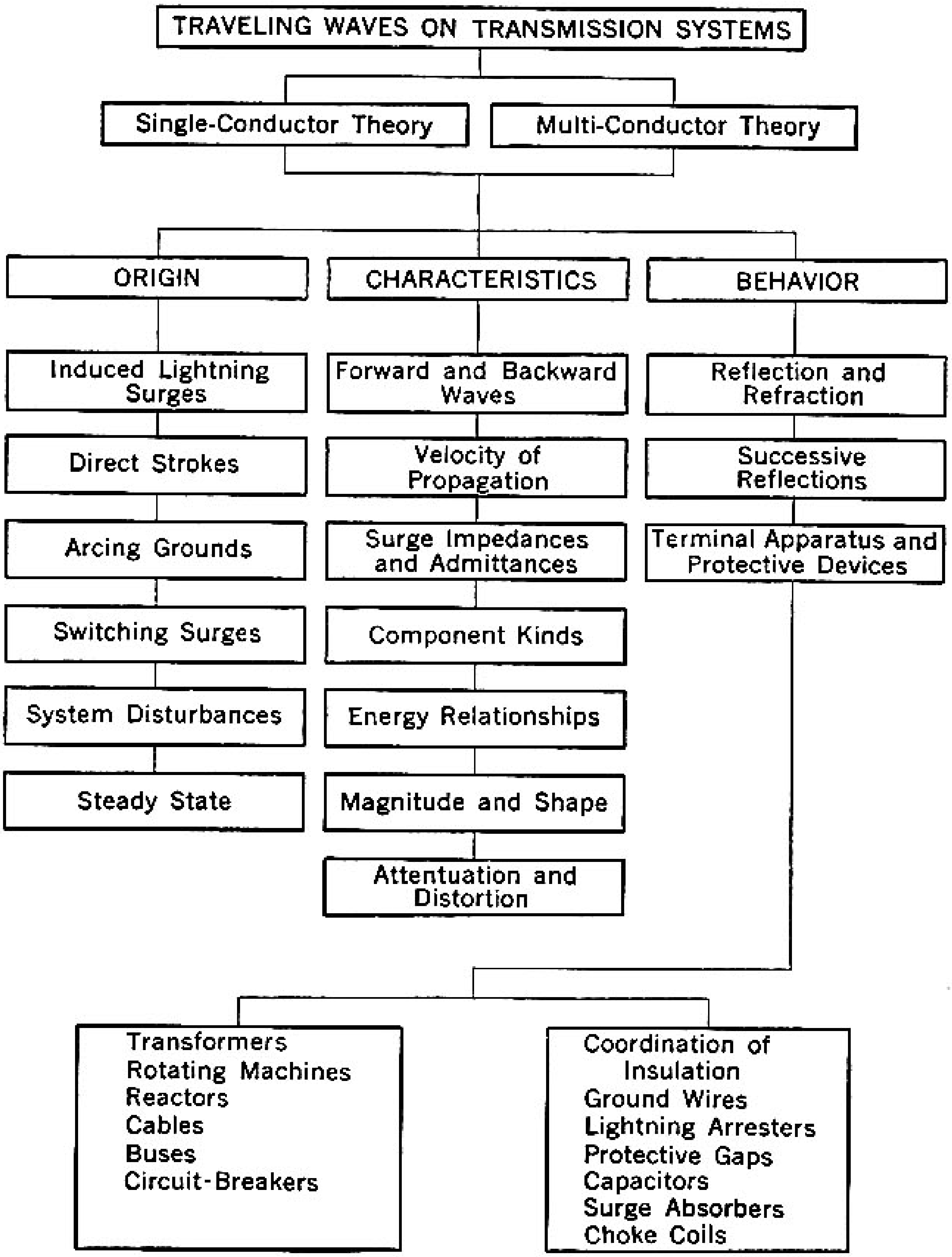
Field studies likewise are in two divisions: the experimental and the statistical. Experimental work in the field has followed the same practice as in the laboratory. In some cases semi-permanent field laboratories have been installed at strategic points along the transmission line; in other cases portable impulse generators and cathode-ray oscillographs, with their auxiliary equipment, have been mounted on trucks or trailers and transported to whatever location was convenient to carry out the specific mission. There was even one instance in which a railway car was fitted up with a cathode-ray oscillograph and living quarters, and used as a mobile laboratory for the purpose of investigating abnormal voltage conditions on the power lines paralleling an electrified railroad.

Statistical studies in the field include such meteorological observations as the number, severity, and paths of lightning storms, the



heights and apparent size of clouds, and photographs of lightning strokes. Many operating companies keep detailed records of all circuit interruptions and carefully correlate these data with the known, or probable, cause. In this way the number of outages due to lightning is known over a period of years, and the value of any protective equipment which is installed in the interim can be evaluated accordingly. Incidentally, the record of apparatus failure kept by operating companies becomes something of a criterion of its reliability.

The theoretical analysis of system disturbances, with special reference to lightning, is the main purpose of this book. There might also be classified under this heading those various and sundry speculations concerning the mechanism of a lightning discharge, the formation of a thunder cloud, etc., but the mathematical derivations of this book do not rest on such opinions. The greater part of the



study of system disturbances involves the theory of traveling waves and is concerned with their origin, characteristics, and behavior at a point of circuit discontinuity. For most purposes the "single-conductor" theory of traveling waves—which considers only a single conductor with ground return—is adequate. But there are cases where it is absolutely necessary to take cognizance of the existence of neighboring conductors, and to consider the mutual reactions between them. The chart given on page 5 of this discussion is a classification of the theory of traveling waves and constitutes an outline of the scope of this book. In Chapter I the basic laws of the single-conductor theory are derived. In Chapter II these laws are applied to a number of typical cases. In Chapter III the subject of attenuation and distortion is discussed and some semi-empirical attenuation formulas are derived. In Chapter IV the principle of superposition is invoked to establish a very simple means for calculating successive (or repeated) reflections. In Chapter V various schemes for the control or annihilation of destructive traveling waves are discussed. These first five chapters end the part devoted to single-conductor theory. Chapter VI derives the general laws of the multi-conductor system. Chapter VII shows how to calculate the behavior of waves at a transition point, and Chapter VIII introduces a resolution into "kinds" of waves which is analogous to symmetrical components in polyphase systems. In Chapters IX and X the theory of traveling waves is applied to the calculation of disturbances caused by lightning. The final Chapter XI gives a résumé of the theory of arcing grounds and switching surges, and terminates Part I of this book.

CHAPTER I

SINGLE-CIRCUIT THEORY OF TRAVELING WAVES

In the single-circuit theory of traveling waves, it is assumed that the phenomenon is confined to a single pair of conductors and the surrounding space. One of these two conductors may be, and usually is, the image of the other in the equipotential plane of the earth surface. In ordinary soil it appears to be sufficient to regard the surface of the ground as a true zero potential plane as far as traveling waves are concerned, although it should not be inferred that such an assumption is valid for calculating telephone interference from the zero phase sequence and triple harmonic currents of transmission systems.

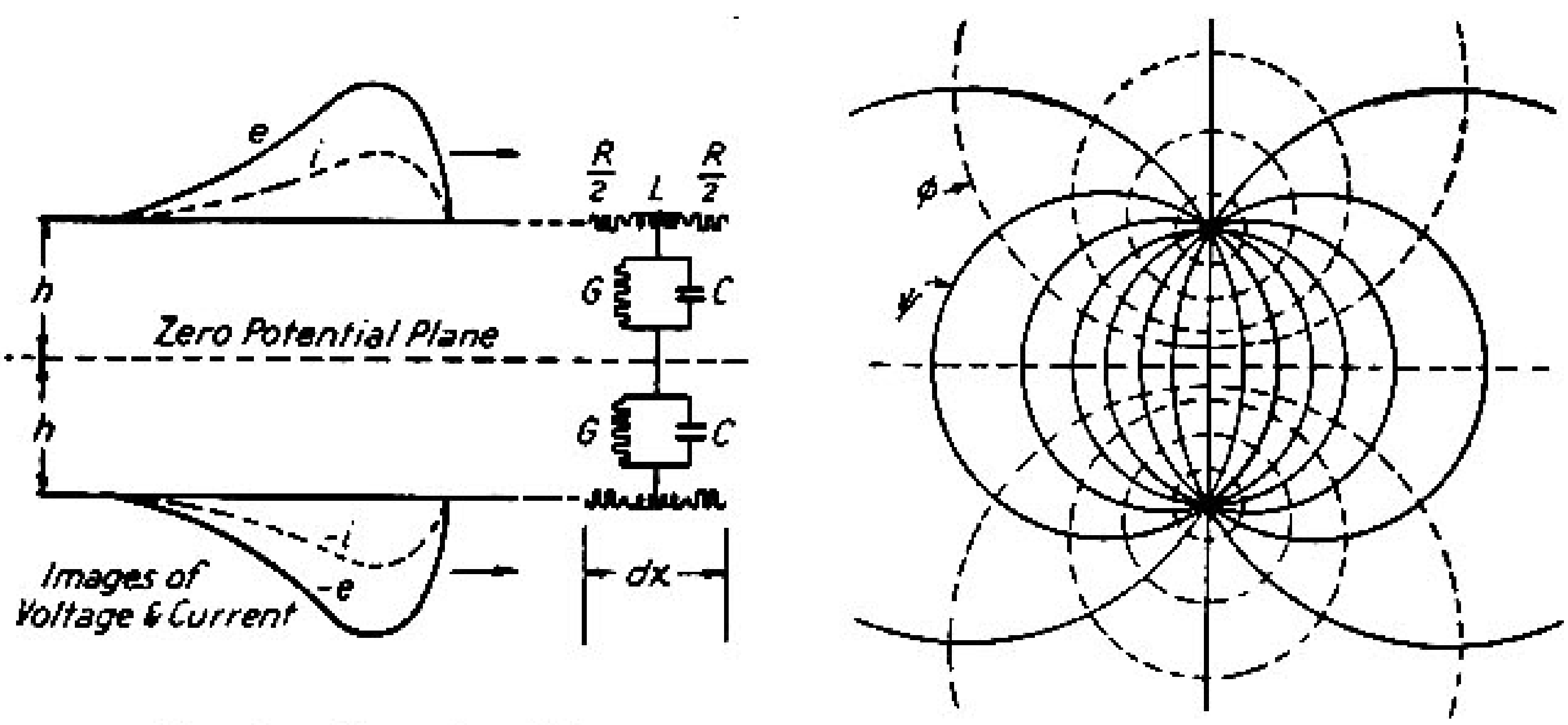


FIG. 1.—Traveling Waves and Associated Electromagnetic Fields

Fig. 1 shows a pair of traveling waves on a single circuit of two conductors. Associated with the voltage wave e there is an electrostatic flux ψ and with the current wave i an electromagnetic flux ϕ, such that for each element dx of the line

$$d\phi = i\,Ldx \tag{1}$$

$$d\psi = e\,Cdx \tag{2}$$

where L is the inductance and C the capacitance per unit length of circuit, to the zero potential plane. The voltage drop in the element dx due to $d\phi$ is $-\partial(d\phi)/\partial t$, to which must be added the resistance drop $-i\,Rdx$, to give the total drop in the direction of positive x, so that

$$-de = -\frac{\partial e}{\partial x}dx = i\,Rdx + \frac{\partial}{\partial t}(d\phi) = \left(R + L\frac{\partial}{\partial t}\right) i\,dx \qquad (3)$$

The charging current of the element dx is $-\partial(d\psi)/\partial t$, to which must be added the leakage current $-e\,Gdx$, to give the total change of current in the direction of positive x, so that

$$-di = -\frac{\partial i}{\partial x}dx = e\,Gdx + \frac{\partial}{\partial t}(d\psi) = \left(G + C\frac{\partial}{\partial t}\right) e\,dx \qquad (4)$$

Canceling dx from both sides of (3) and (4) there is

$$-\frac{\partial e}{\partial x} = \left(R + L\frac{\partial}{\partial t}\right) i = Z(p)\cdot i \qquad (5)$$

$$-\frac{\partial i}{\partial x} = \left(G + C\frac{\partial}{\partial t}\right) e = Y(p)\cdot e \qquad (6)$$

in which $p = \partial/\partial t$ is the time derivative. Equations (5) and (6) are the well-known differential equations of the single-circuit transmission line. Differentiating (5) with respect to x and substituting (6), there is

$$\frac{\partial^2 e}{\partial x^2} = -\,Z(p)\cdot\frac{\partial i}{\partial x} = Y(p)\cdot Z(p)\,e$$

$$= [RG + (RC + GL)\,p + LC\,p^2]\,e \qquad (7)$$

Or differentiating (6) with respect to x and substituting (5), there is

$$\frac{\partial^2 i}{\partial x^2} = -\,Y(p)\,\frac{\partial e}{\partial x} = Z\,(p)\cdot Y\,(p)\,i$$

$$= [RG + (RC + GL)\,p + LC\,p^2]\,i \qquad (8)$$

Thus it is seen that the differential equation is the same for either the voltage or the current, and therefore the solutions for e and i will differ only in the terminal and initial conditions. Solutions to this equation—which mathematical physicists refer to as "the telegraph equation"—were obtained many years ago by Heaviside in England and by Poincaré in France, and today it is dealt with quite commonly in electrical engineering literature.* Solving (7) and (8) as ordinary differential equations in x

$$e = \varepsilon^{x\sqrt{ZY}} f_1(t) + \varepsilon^{-x\sqrt{ZY}} f_2(t) \tag{9}$$

$$i = -Y\int e dx = -\sqrt{\frac{Y}{Z}}[\varepsilon^{x\sqrt{ZY}} f_1(t) - \varepsilon^{-x\sqrt{ZY}} f_2(t)] \tag{10}$$

where $f_1(t)$ and $f_2(t)$ are integration constants with respect to x only, and are therefore possible functions of t. Equations (9) and (10) are general operational solutions to (7) and (8), but in this form are not readily evaluated because of the radical

$$\sqrt{ZY} = \sqrt{LC\left[\frac{RG}{LC} + \left(\frac{R}{L} + \frac{G}{C}\right)p + p^2\right]} \tag{11}$$

Heaviside noticed that if $RC = GL$ then (11) simplifies to

$$\sqrt{ZY} = \sqrt{LC\left(p + \frac{R}{L}\right)^2} = \sqrt{LC}\left(p + \frac{R}{L}\right) \tag{12}$$

and if the losses are negligible

$$\sqrt{ZY} = \sqrt{LC}\, p \tag{13}$$

Now by Taylor's theorem

$$\begin{aligned} f(t+a) &= f(t) + a f'(t) + \frac{a^2}{\underline{|2}} f''(t) + \dots \\ &= (1 + ap + \frac{a^2}{\underline{|2}} p^2 + \dots) f(t) \\ &= \varepsilon^{ap} f(t) \end{aligned} \tag{14}$$

Comparing (9) and (10) with (14) it is evident that if (12) holds

$$e = \varepsilon^{\sqrt{CL}\,(R/L)x} f_1(t + \sqrt{CL}\,x) + \varepsilon^{-\sqrt{CL}\,(R/L)x} f_2(t - \sqrt{CL}\,x) \tag{15}$$

$$i = -\sqrt{\frac{C}{L}}[\varepsilon^{\sqrt{CL}\,(R/L)x} f_1(t + \sqrt{CL}x) - \varepsilon^{-\sqrt{CL}\,(R/L)x} f_2(t - \sqrt{CL}\,x)] \tag{16}$$

and if the losses are negligible, $R = 0$,

$$e = f_1(t + \sqrt{CL}\,x) + f_2(t - \sqrt{CL}\,x) \tag{17}$$

$$i = -\sqrt{\frac{C}{L}}[f_1(t + \sqrt{CL}\,x) - f_2(t - \sqrt{CL}\,x)] \tag{18}$$

* See, for example, Steinmetz's "Transient Electric Phenomena and Oscillations."

Now $f(t \pm \sqrt{CL}\, x)$ represents traveling waves, because for any t a corresponding x can be found such that $f(t \pm \sqrt{CL}\, x)$ has the same value. Corresponding values of x and t which define the same point on the wave are given by

$$\left.\begin{aligned} t - \sqrt{CL}\, x &= \sqrt{CL}\, \lambda_2 \text{ for the forward wave} \\ t + \sqrt{CL}\, x &= \sqrt{CL}\, \lambda_1 \text{ for the reverse wave} \end{aligned}\right\} \tag{19}$$

The velocity of propagation is

$$\left.\begin{aligned} v &= \frac{dx}{dt} = \frac{+1}{\sqrt{CL}} \text{ for the forward wave} \\ v &= \frac{dx}{dt} = \frac{-1}{\sqrt{CL}} \text{ for the reverse wave} \end{aligned}\right\} \tag{20}$$

Thus the voltage and current distributions are propagated as traveling waves, and each may consist of a forward wave f_2 moving in the direction of positive x, and a reverse wave f_1 moving in the direction of negative x, and both waves have the same velocity $v = 1/\sqrt{CL}$.

For parallel wires in air whose distance apart is large compared to their radii, the constants per wire are

$$L = \left(\frac{1}{2} + 2 \log_e \frac{2\,h}{r}\right) 10^{-9} \text{ henry per cm.} \tag{21}$$

$$C = \frac{10^{-11}}{\left(18 \log_e \frac{2\,h}{r}\right)} \text{ farads per cm.} \tag{22}$$

where $2\,h$ is the spacing between conductors and r is the radius of each conductor. Therefore, neglecting the factor 1/2 in (21)

$$v \cong \frac{1}{\sqrt{LC}} = 3 \times 10^{10} \text{ cm. per sec.} = 985 \text{ ft. per microsecond}$$

$$= \text{velocity of light in free space.} \tag{23}$$

In traveling-wave literature, t is usually measured in microseconds (millionths of a second) and x is measured in thousands of feet. Then v is approximately unity.

In the case of a cable with a solid inner conductor of radius r, inside sheath radius R, and insulation of permittivity k,

$$L = \left(2 \log_e \frac{R}{r} + \frac{1}{2} + \frac{r^2}{3\,R^2} - \frac{r^4}{12\,R^4} + \frac{r^6}{30\,R^6} - \cdots\right) 10^{-9} \quad (24)$$

$$C = \frac{k\,10^{-11}}{18 \log_e \frac{R}{r}} \quad (25)$$

A very rough approximation for v based on ignoring all except the first term in L is

$$v \cong \frac{3 \times 10^{10}}{\sqrt{k}} \text{ cm. per sec.} \quad (26)$$

In most commercial cables, k varies from $k = 2.5$ to $k = 4$, so that the velocity of propagation in a cable is from one-half to two-thirds that of light.

Substituting (19) into (15) and (16) respectively

$$e = \varepsilon^{(-R/L)\,t}\left\{\varepsilon^{(R/L)\sqrt{CL}\,\lambda_1} f_1(\sqrt{CL}\,\lambda_1) + \varepsilon^{-(R/L)\sqrt{CL}\,\lambda_2} f_2(\sqrt{CL}\,\lambda_2)\right\} \quad (27)$$

$$= \varepsilon^{(-R/L)\,t}\left\{f_3\,(\lambda_1) + f_4(\lambda_2)\right\}$$

$$= \varepsilon^{(-R/L)\,t}\left\{f_3\,(x + vt) + f_4\,(x - vt)\right\} \quad (28)$$

$$i = \varepsilon^{(-R/L)\,t}\left\{-f_3\,(x + vt) + f_4\,(x - vt)\right\}\sqrt{\frac{C}{L}} \quad (29)$$

Thus the decrement factor may be a *space* decrement as in (15) and (16), or a *time* decrement as in (28) and (29). At the initial instant, $t = 0$

$$e = \varepsilon^{\sqrt{CL}\,(R/L)\,x} f_1(x) + \varepsilon^{-\sqrt{CL}\,(R/L)\,x} f_2(x)$$

$$= f_3(x) + f_4(x) \quad (30)$$

It is therefore evident that, if the voltage distribution along the conductor is specified at $t = 0$, it is more convenient to use (28) than (15), for then the shape of the wave is the same as the initial distribution; whereas if (15) is used it is necessary to divide the ordinates of the initial distribution through by the exponential factor in order to find the wave shape. On the other hand, if the shape of the wave is specified as a function of time at $x = 0$ it is more convenient to use (15) so as to avoid a division by the exponential Rt/L in order to obtain the wave shape.

From (28) and (29) it is seen that the following relationship exists between corresponding potential and current waves:

$$\left.\begin{aligned} e &= +\,i\sqrt{L/C} \quad \text{or} \quad i = e\sqrt{C/L} \quad \text{for forward waves} \\ e &= -\,i\sqrt{L/C} \quad \text{or} \quad i = -\,e\sqrt{C/L} \text{ for reverse waves} \end{aligned}\right\} \tag{31}$$

The quantity $\sqrt{L/C} = z$ is called the *surge impedance* of the circuit. Its reciprocal, $\sqrt{C/L} = y$, is the *surge admittance*. They are measured directly in *ohms* and *mhos* respectively, because they have the dimensions of resistance and conductance. It will be noticed that the per-unit-length factor does not enter into their description.

It is interesting to note the following identities:

$$z = \frac{1}{y} = \sqrt{\frac{L}{C}} = vL = \frac{1}{vC} \tag{32}$$

Thus if L is known, then $C = 1/v^2L$ and $z = vL$. These relationships are of practical importance in many cases where, for one reason or another, it is difficult to measure certain constants.

The fact that the solutions obtained in (28) and (29) contain both a forward and a reverse wave does not mean that both waves must actually be present. Either wave by itself is a solution of the differential equation, and if a single wave satisfies the line conditions, then the other wave is unnecessary in the solution. Nor should it be supposed that a single forward and a single reverse wave will satisfy all line conditions. More generally, (28) and (29) satisfy the differential equation for any functions f_3 and f_4, and therefore all such functions satisfy the differential equation, so that the complete solution takes the form

$$e = \varepsilon^{-(R/L)t}\Sigma[f_3(x + vt) + f_4(x - vt)] \tag{33}$$

$$i = y\varepsilon^{-(R/L)t}\Sigma[-f_3(x + vt) + f_4(x - vt)] \tag{34}$$

These solutions are based on the premise that $RC = GL$. A line for which this condition holds is called the Heaviside *distortionless line*. This, and the absence of losses, are the only conditions under which pure waves can exist on a transmission line. It is telephone engineering practice to add loading coils to the line at intervals so as to approximate the distortionless condition $RC = GL$ and thereby minimize such distortion.

On a power-transmission circuit, distortion is always present. Nevertheless, experience has demonstrated that lightning surges may be treated as pure waves, and that the decrement may be taken care of by an external factor, such as in Equation (32), called the attenuation factor. The conventional transmission-line theory supposes the line parameters R, L, G, C, to be true constants, but as a matter of fact they vary over a wide range depending upon the voltage and shape of the surge. Thus the resistance R increases and the inductance L decreases as the steepness of the wave becomes more abrupt, because the transient skin effect drives the current from the inner part of the conductor. Both the conductance G and the capacitance C increase with the formation of corona, when the critical disruptive corona voltage is exceeded. The law by which this change in G and C takes place has not yet been established, and even if it were known, there is no reason to suppose that the transmission-line differential equations could then be solved, or if a solution was found that it could be used in engineering calculations. For these reasons it is not felt that a discussion of wave distortion, based on the solution for constant values of R, L, C, and G, is worth while from the point of view of this book.

The energy content of a corresponding pair of potential and current waves is, making use of (31),

$$
\begin{aligned}
W = W_e + W_i &= \frac{C}{2}\int e^2 dx + \frac{L}{2}\int i^2 dx \\
&= \frac{C}{2}\int e^2 dx + \frac{L}{2}\int \frac{C}{L} e^2 dx \\
&= C\int e^2 dx = L\int i^2 dx = \sqrt{LC}\int ei\, dx \qquad (35)
\end{aligned}
$$

Or integrating with respect to time

$$
W = \int ei\, dt = \sqrt{\frac{C}{L}}\int e^2 dt = \sqrt{\frac{L}{C}}\int i^2 dt \qquad (36)
$$

In both (35) and (36) the integration is to include the entire wave length. Equations (35) and (36) are continuations of each other through the relationship $v dt = dx$ from (20). These equations show that the total energy of a pair of traveling waves is divided equally between the potential and the current waves, for

$$
W_e = \frac{C}{2}\int e^2 dx = \frac{W}{2}
$$

$$
W_i = \frac{L}{2}\int i^2 dx = \frac{W}{2}
$$

However, when a pair of traveling waves reach a transition point, or when waves traveling in the opposite direction pass through each other, the energy balance is upset, and more energy will reside in one field than in the other. Equations (35) and (36), however, give the correct energy distribution for each component wave.

Behavior of a Traveling Wave at a Transition Point.—When a traveling wave reaches a transition point at which there is an abrupt change of circuit constants, as an open- or short-circuited terminal, or a junction with another line, etc., a part of the wave is reflected back, and a part may pass on to other sections of the circuit. The impinging wave is called an *incident* wave, and the two waves to which it gives rise at a transition point are called the *reflected* and *transmitted* waves respectively. Such waves are formed at the transition point in accordance with Kirchhoff's laws. They satisfy the differential equations of the transmission line, and are consistent with the principle of the conservation of energy.

Suppose that the line is closed at the transition point by a general impedance consisting of any arrangement of inductances, resistances, capacitances, and other lines. Let the operational equation specifying this general impedance be written as $Z_0(p)$. Let the transition point be taken as the origin of coordinates, and distance along the line away from the point be counted as negative, so that an approaching wave is traveling in the positive direction. By Equation (31) the potential and current incident waves will then have the same sign. Denote the incident waves by e and i, the reflected waves by e' and i', and the transmitted waves, if they exist, by e'' and i''. Then the total potential at the transition point is, using (31),

$$e_0 = e + e' = (i + i')\,Z_0\,(p) = (e - e')\,y\,Z_0\,(p) \tag{37}$$

where

$$y = 1/z = \sqrt{C/L} = \text{surge admittance.}$$

Solving this equation for e' there is

$$e' = \left[\frac{Z_0\,(p) - z}{Z_0\,(p) + z}\right] e = \text{reflected potential wave} \tag{38}$$

The total resultant wave at the transition point is the sum of the incident and reflected waves,

$$e_0 = e + e' = \left[\frac{2\,Z_0\,(p)}{Z_o\,(p) + z}\right] e \tag{39}$$

$$i_0 = i + i' = y\,(e - e') = \left[\frac{2}{Z_0\,(p) + z}\right] e \tag{40}$$

In general, $Z_0(p)$ may consist of any number of branches in parallel. One of the most general types of transition points met with in transmission systems consists of a junction at which there is a general impedance network to ground $Z_g(p)$; and n transmission lines of surge impedances $(z_1, \ldots z_n)$ joined through networks $Z_1(p) \ldots Z_n(p)$, respectively. Such a system is shown in Fig. 2. When an incident wave e_1 approaching along the line z_1 reaches the junction, it will give rise to a wave e_1' reflected back on line z_1; transmitted

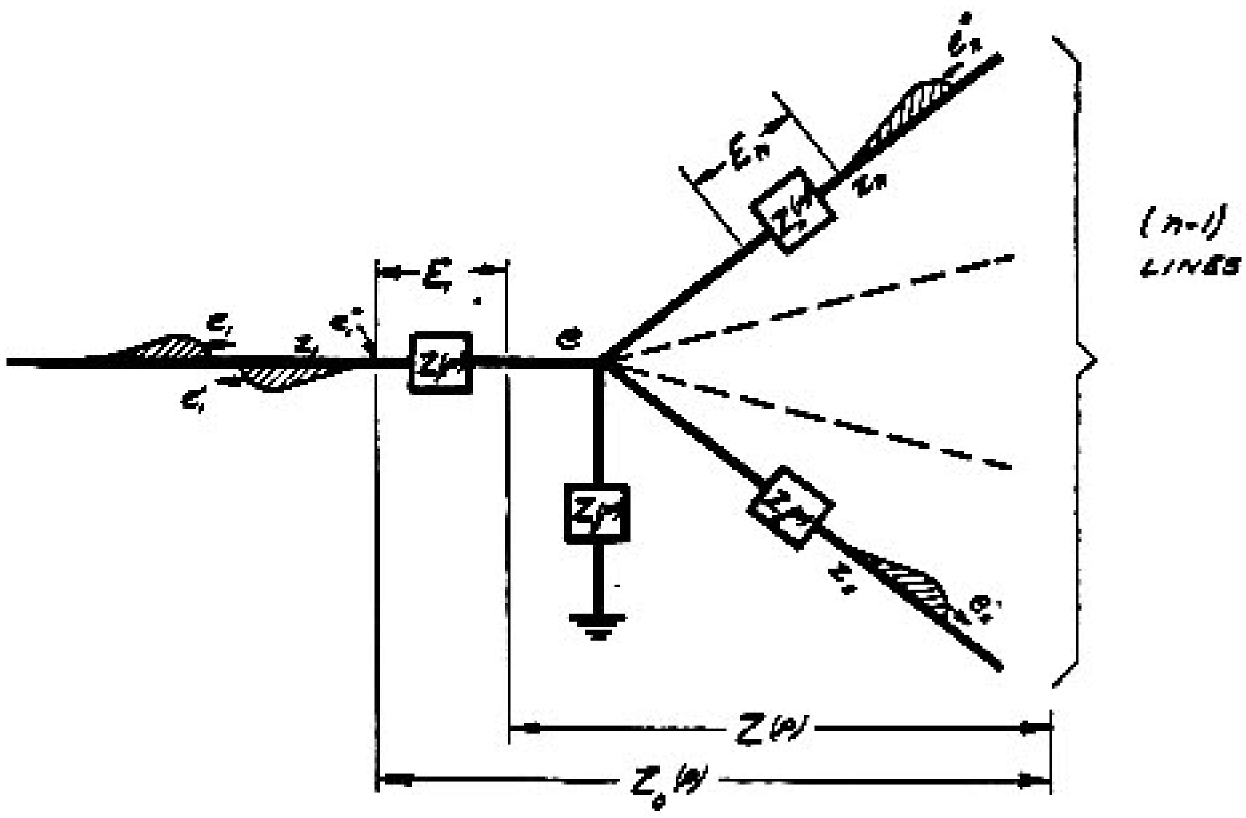


FIG. 2.—General Transition Point

waves $e_2'', \ldots e_n''$ on lines z_2 to z_n, respectively; and a potential e at the junction. Now the total impedance at the end of z_1 is

$$Z_0(p) = Z_1(p) + Z(p) = Z_1(p) + \frac{1}{\dfrac{1}{Z_g(p)} + \displaystyle\sum_2^n \frac{1}{z_k + Z_k(p)}} \tag{41}$$

and the reflected potential wave on z_1 therefore is

$$e_1' = \left[\frac{Z_0(p) - z_1}{Z_0(p) + z_1}\right] e_1 \tag{42}$$

The total potential at the reflection point is

$$e_1'' = (e_1 + e_1') = \left[\frac{2\,Z_0(p)}{Z_0(p) + z_1}\right] e_1 \tag{43}$$

The current transmitted across $Z_1\,(p)$ is

$$i_1'' = \frac{e_1''}{Z_0\,(p)} = \left[\frac{2}{Z_0\,(p) + z_1}\right] e_1 \tag{44}$$

The potential at the grounding impedance network is

$$e = \frac{Z\,(p)}{Z_0\,(p)}\, e_1'' = \left[\frac{2\,Z\,(p)}{Z_0\,(p) + z_1}\right] e_1 \tag{45}$$

and the current through $Z_g\,(p)$ therefore is

$$i_g = \frac{e}{Z_g\,(p)} = \frac{Z\,(p)}{Z_0\,(p)}\,\frac{e_1''}{Z_g\,(p)}$$

$$= \left[\frac{2\,Z\,(p)}{Z_0\,(p) + z_1}\right] \frac{e_1}{Z_g\,(p)} \tag{46}$$

The current and voltage waves transmitted to any line k (where $2 \leq k \leq n$) are

$$i_k'' = \frac{e}{Z_k\,(p) + z_k} = \frac{Z\,(p)}{Z_0\,(p)}\,\frac{e_1''}{Z_k\,(p) + z_k}$$

$$= \left[\frac{2\,Z\,(p)}{Z_0\,(p) + z_1}\right] \frac{e_1}{Z_k\,(p) + z_k} \tag{47}$$

$$e_k'' = z_k i_k'' = \frac{Z\,(p)}{Z_0\,(p)}\,\frac{z_k e_1''}{Z_k\,(p) + z_k}$$

$$= \left[\frac{2\,Z\,(p)}{Z_0\,(p) + z_1}\right] \frac{z_k e_1}{Z_k\,(p) + z_k} \tag{48}$$

The potential drops across the lumped impedance networks are

$$E_1 = e_1'' - e = \frac{Z_1\,(p)}{Z_0\,(p)}\, e_1'' = \frac{2\,Z_1\,(p)}{Z_0\,(p) + z_1}\, e_1 \tag{49}$$

$$E_k = e - e_k'' = \left[\frac{Z_k\,(p)}{Z_k\,(p) + z_k}\right] \frac{Z\,(p)}{Z_0\,(p)}\, e_1''$$

$$= \left[\frac{Z_k\,(p)}{Z_k\,(p) + z_k}\right] \frac{2\,Z\,(p)}{Z_0\,(p) + z_1}\, e_1 \tag{50}$$

Thus if e_1 is known at the transition point as a function of time, then the other voltages and currents are determined by solving the above differential equations. In particular, if e_1 is a rectangular wave with an infinite tail, it may be taken as Heaviside's *unit function*

A, and the solution obtained by means of operational calculus. The solution for a finite wave of any shape may then be found from Duhamel's theorem, Equation (36) of the Appendix.

The division of energy during the transition period at the junction furnishes a valid check in any specific case on the above relationships, and is of interest on its own account. At any time t, counting from the instant that the incident wave e_1 arrives at the junction, there is (dropping the p's for simplicity)

$$\int_t^\infty e_1 \left(\frac{e_1}{z_1}\right) dt = \text{energy remaining in the incident wave} \tag{51}$$

$$\int_0^t e_1' \left(\frac{e_1'}{z_1}\right) dt = \text{energy in the reflected wave} \tag{52}$$

$$\int_0^t e_k'' \left(\frac{e_k''}{z_k}\right) dt = \text{energy in the wave transmitted to line } k \tag{53}$$

$$\int_0^t E_k \left(\frac{E_k}{Z_k}\right) dt = \text{energy absorbed by impedance network } Z_k\,(p) \tag{54}$$

$$\int_0^t e \left(\frac{e}{Z_g}\right) dt = \text{energy absorbed by impedance network } Z_g\,(p) \tag{55}$$

Equating to the energy of the original incident wave, by the conservation of energy,

$$\begin{aligned}\int_0^\infty e_1 \left(\frac{e_1}{z_1}\right) dt = \int_t^\infty e_1 \left(\frac{e_1}{z_1}\right) dt + \int_0^t e_1' \left(\frac{e_1'}{z_1}\right) dt \\ + \sum_2^n \int_0^t \left[e_k'' \left(\frac{e_k''}{z_k}\right) + E_k \left(\frac{E_k}{Z_k}\right) \right] dt \\ + \int_0^t E_1 \left(\frac{E_1}{Z_1}\right) dt + \int_0^t e \left(\frac{e}{Z_g}\right) dt \end{aligned} \tag{56}$$

But the two terms involving e_1 may be combined as

$$\begin{aligned}\int_0^\infty e_1 \left(\frac{e_1}{z_1}\right) dt - \int_t^\infty e_1 \left(\frac{e_1}{z_1}\right) dt \\ = \int_0^t e_1 \left(\frac{e_1}{z_1}\right) dt \end{aligned} \tag{57}$$

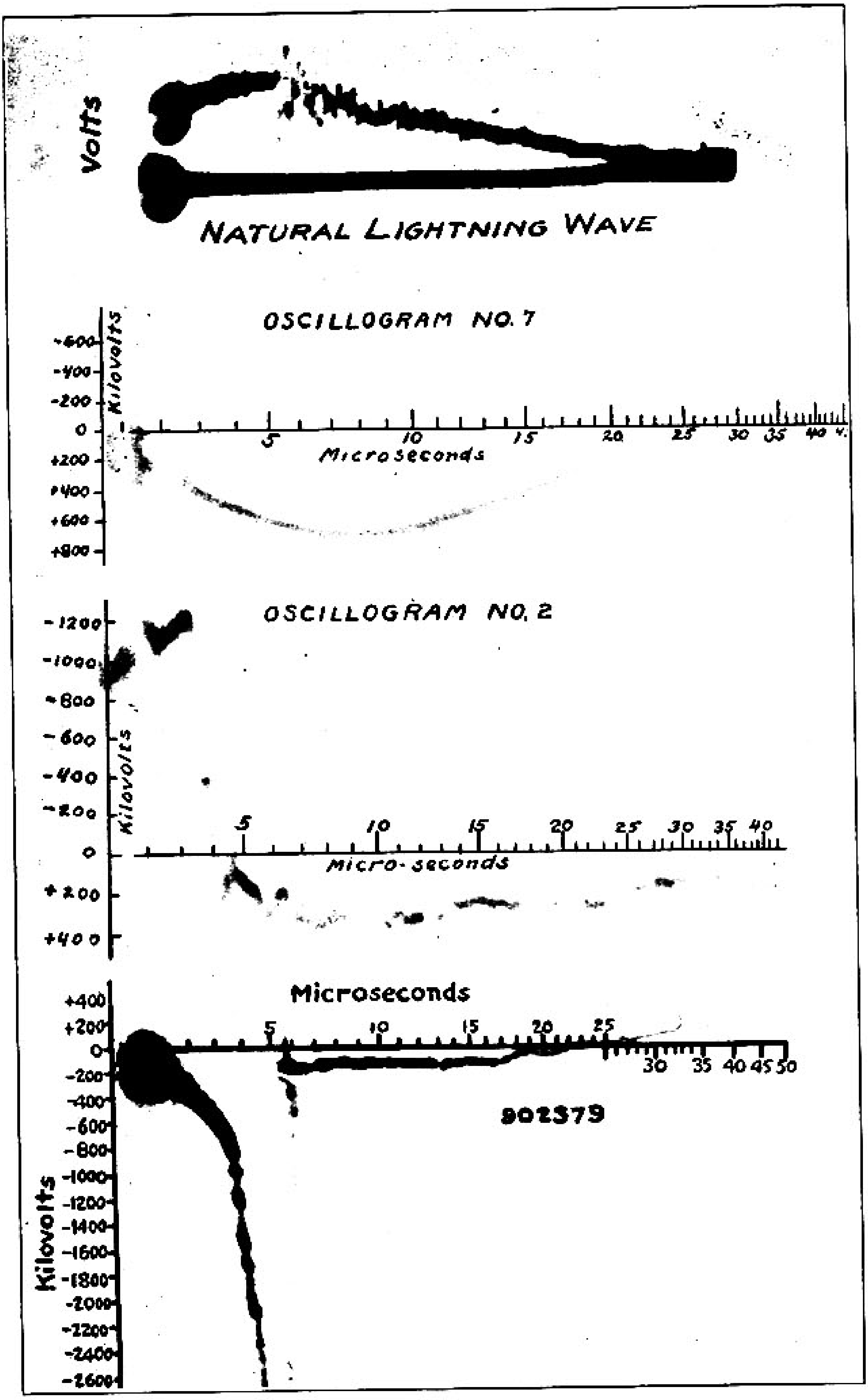


FIG. 3.—Cathode Ray Oscillograms of Typical Natural Lightning Waves

Then, discarding the integrals, there is the general relationship

$$\frac{e_1^2}{z_1} = \frac{e_1'^2}{z_1} + \sum_2^n \left[e_k''\left(\frac{e_k''}{z_k}\right) + E_k\left(\frac{E_k}{Z_k}\right)\right] + E_1\left(\frac{E_1}{Z_1}\right) + e\left(\frac{e}{Z_g}\right) \tag{58}$$

Substituting throughout, in terms of e_1, from the previous equations of reflection and refraction, this expression reduces to an identity, thus proving that the reflection and refraction operators are consistent with the energy relationships.

Shape and Specification of Traveling Waves.—An examination of the cathode-ray oscillograms of natural lightning waves which have been obtained during the past few years, Fig. 3, discloses that the most of them are of relatively simple shape, although nearly all of them are serrated by minor irregularities. The principal features of the wave shapes caused by natural lightning are included in Fig. 4, that is, may be represented by the difference of two exponentials. This is an extremely fortunate situation, since calculations of the effect of traveling waves on terminal equipment are more easily carried out for exponential components than for any other wave shape, except the infinite rectangular wave. So many of the cathode-ray oscillograms have the same general outline as Fig. 4*D* that it is probably permissible to speak of that shape as the typical lightning wave.

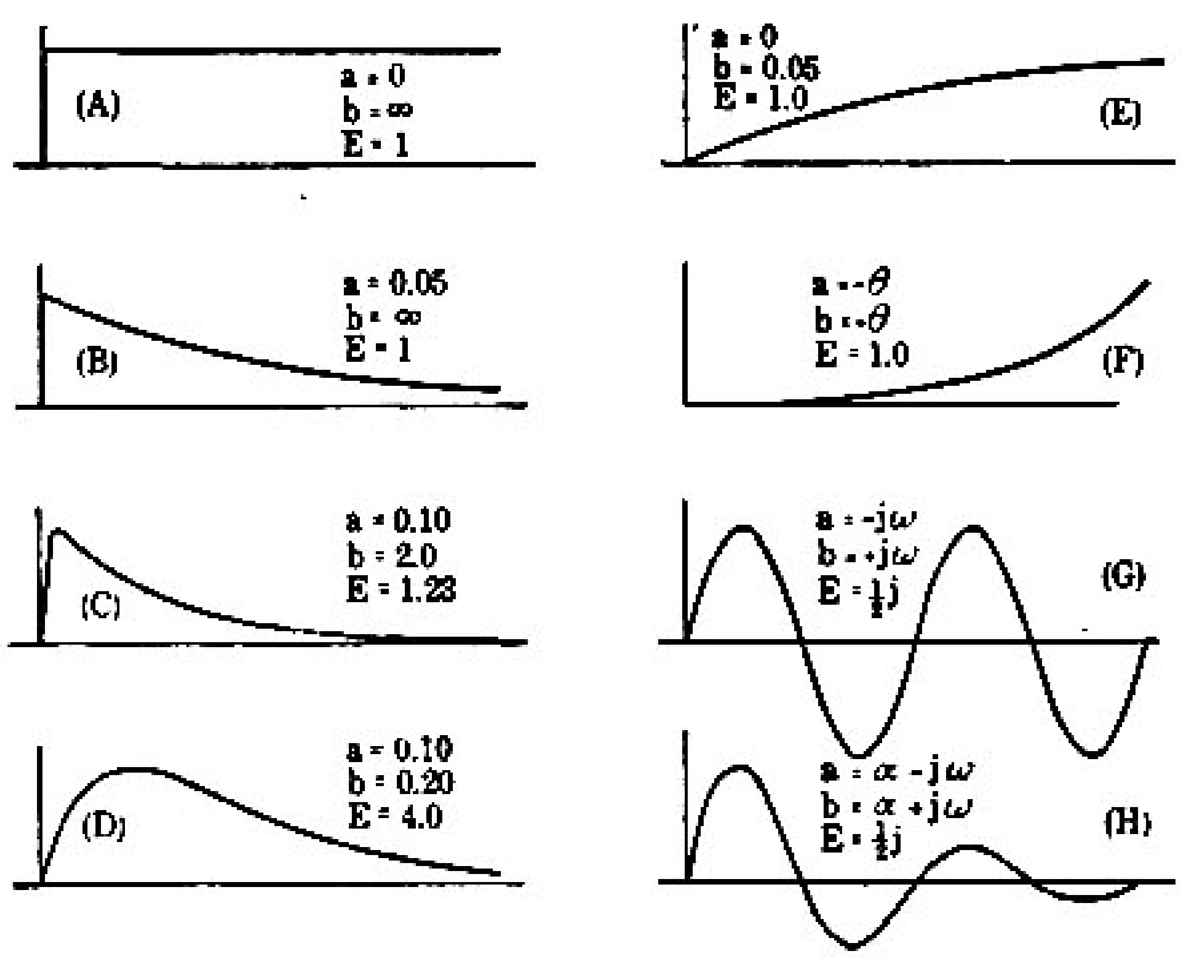


FIG. 4.—Empirical Wave Shapes Given by
$e = E\,(\varepsilon^{-at} - \varepsilon^{-bt})$

A traveling wave is characterized by four specifications. The *crest* of the wave is its maximum amplitude. The *front* is that part of the wave from the beginning to the crest. The *tail* is that portion of the wave behind the crest. The *polarity* of the wave is the polarity of the crest, or in the case of an oscillatory wave, is given as the polarities of successive loops. Practically, in designating a wave, it has become customary to disregard that part of the tail beyond the point at which the wave has decreased to half value. Thus in speaking of a wave as 20 ms. long it is understood that the length of the wave to the 50 per cent value on the tail is 20 ms. For brevity, a 750-kv., positive polarity wave with a 3.5-ms. front and a 22-ms. tail (to the 50 per cent point) may be designated by 750 kv., 3.5/22/+.

As far as mathematical simplicity is concerned, the simplest wave to calculate the effects of is the infinite rectangular, shown in Fig. 4*A*. Also, as a rule, such a wave is the most dangerous to terminal equipment, and therefore calculations based on it are apt to err on the side of safety. Still other reasons that have favored its use in analysis are that it is by far the easiest to study pictorially, and that until recently the actual shapes of lightning surges were not known. However, during the past few years a great many cathode-ray oscillograms of natural lightning waves have been obtained under many different conditions, so that fairly definite information as to their general shape and characteristics is available. It is therefore essential that calculations be made with these characteristic lightning waves, in order that the influence of the fronts, tails, and lengths of the wave may be evaluated.

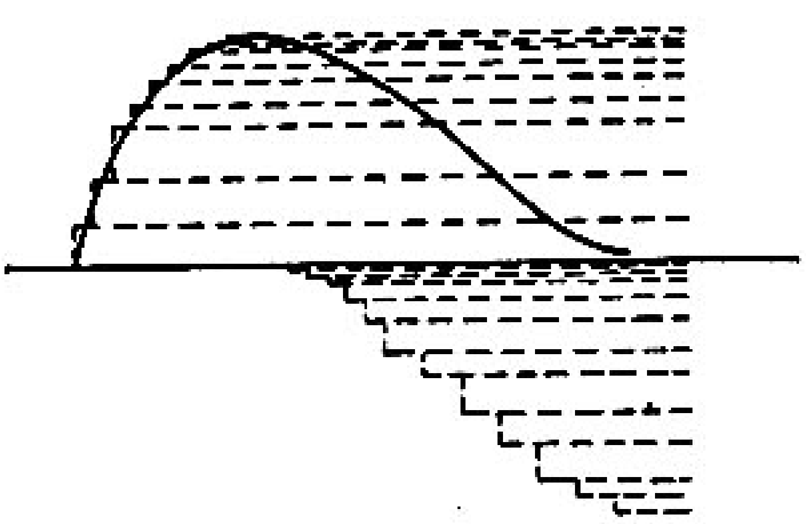


FIG. 5.—Approximation by Rectangular Components

It is always possible to employ the graphical representation of a wave of arbitrary shape as a set of rectangular components, Fig. 5, and the approximation can be made as good as required. The solution corresponding to such a wave is the sum of the solutions corresponding to its rectangular components. In some cases the applied wave may be so complicated as to defy analytic expression, and then a graphical resolution into rectangular components may be the only way out of the difficulty. Duhamel's theorem, Equation (36) of the Appendix, is the analytical counterpart of this method, and is, in fact, the mathematical expression for the principle of superposition.

When a wave shape is too complicated to be represented by the difference of two exponentials, possibly it may be formed by compounding waves as illustrated in Fig. 6. This method of representing the wave as a sum of functions for which the solutions are known is very powerful and practicable. There are a few simple functions for which the response of a network can usually be computed with reasonable ease, and by compounding such functions almost any desired wave shape can be reproduced to a good approximation. Some of these elementary waves are:

a. Infinite rectangular.
b. Simple exponential.
c. Uniformly rising front.
d. Damped sinusoid.
e. Difference of two exponentials.

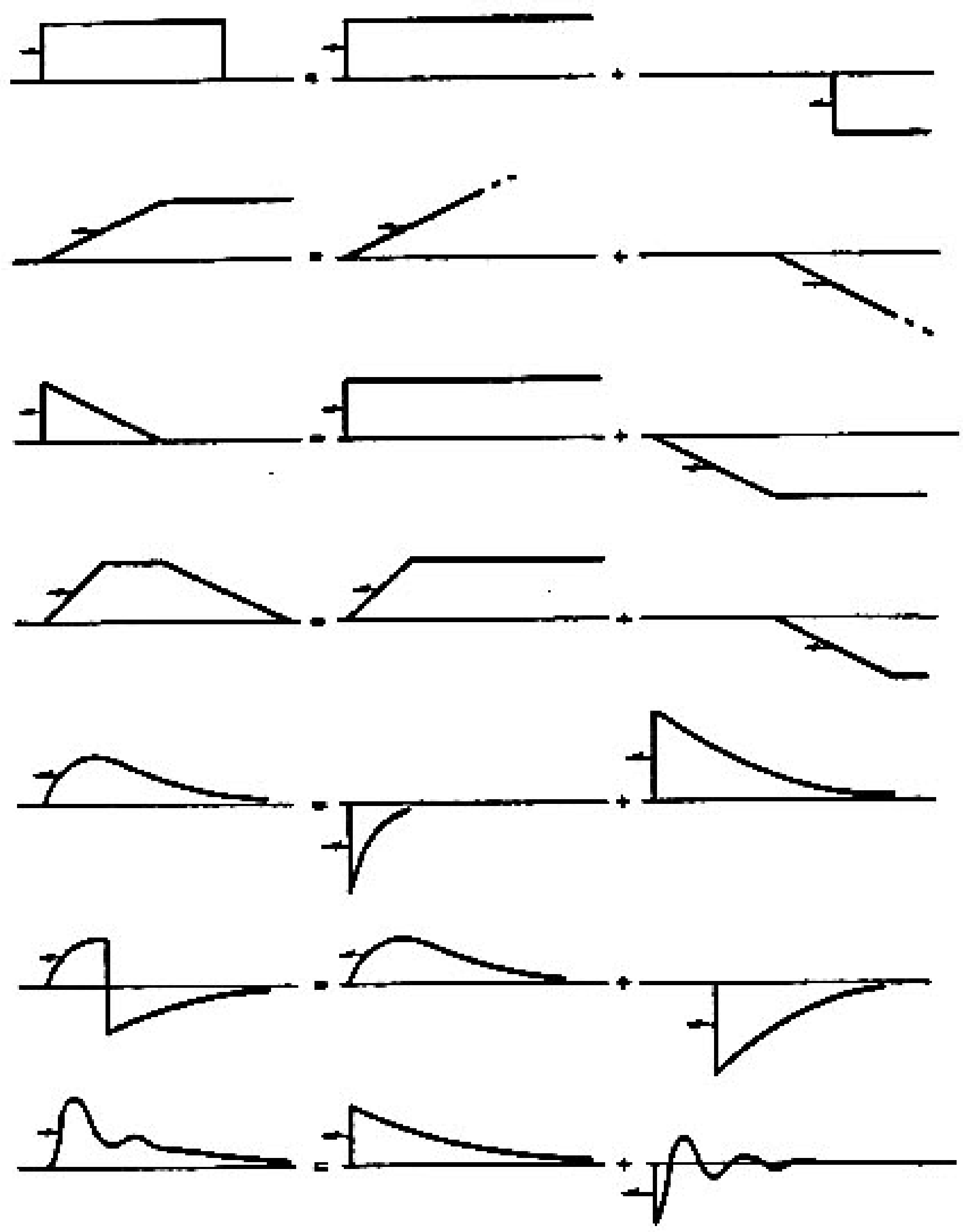


FIG. 6.—Compounding of Simple Waves to Obtain Complex Waves

As a matter of fact, by a suitable choice of the parameters a, b, and E in the differences of two exponentials

$$e = E\,(\varepsilon^{-at} - \varepsilon^{-bt}) \tag{59}$$

all these elementary waves may be considered as special cases of Equation (59). Thus:

If $a = 0$, and $b = \infty$, then $e = E$. The applied wave is infinite rectangular.

If $a = a$, and $b = \infty$, then $e = E\,\varepsilon^{-at}$. This is a wave with an abrupt front and an exponential tail.

If $a = 0$ but $b \to 0$ and $E \to \infty$ in such a way that $(b\,E)$ remains finite, then

$$e = E\,(1-\varepsilon^{-bt}) = -\,E\left(-\,bt + \frac{b^2t^2}{2} - \ldots\right) = b\,E\,t$$

This is a uniformly rising front, or infinite triangular wave.

If $a = (\alpha - j\omega)$ and $b = (\alpha + j\omega)$, and $E = E_0/2\,j$, then

$$e = \frac{E_0}{2j}\,\varepsilon^{-\alpha t}\,(\varepsilon^{j\omega t} - \varepsilon^{-j\omega t}) = E_0\,\varepsilon^{-\alpha t}\sin\omega t$$

which is a damped sinusoidal wave.

If $a = -\,\alpha$ and $b = +\,\alpha$ there results the infinite hyperbolic *sinh* wave

$$e = E\,(\varepsilon^{\alpha t} - \varepsilon^{-\alpha t}) = 2\,E\sinh\alpha t$$

If a and b are both finite and real, then Equation (59) defines a wave with rounded front and exponential tail. It is of interest to examine this case in detail, since it is the most typical of actual lightning surges.

The three parameters a, b, and E, are sufficient to determine uniquely the *crest*, *wave length*, and *front* of the wave. The wave maximum occurs when

$$\frac{de}{dt} = 0 = E\,(-a\varepsilon^{-at} + b\varepsilon^{-bt})$$

hence at that value of t for which

$$t = t_1 = \frac{\log b/a}{b - a} = \frac{1}{a}\left(\frac{\log b/a}{b/a - 1}\right) = \frac{B}{a} \tag{60}$$

and the crest voltage therefore is

$$E_1 = E\,(\varepsilon^{-at_1} - \varepsilon^{-bt_1}) = E\,(\varepsilon^{-B} - \varepsilon^{-B\,(b/a)}) \tag{61}$$

Now from (60) and (61) we may plot at_1 and E_1/E against b/a, since B is a function of b/a only.

The time t_2 at which the wave decreases to half value on the tail is given by

$$\frac{E_1}{2} = E\,(\varepsilon^{-at_2} - \varepsilon^{-bt_2}) = E\,(\varepsilon^{-B\,(t_2/t_1)} - \varepsilon^{-(b/a)\,B\,(t_2/t_1)}) \qquad (62)$$

Hence by (61) and (62)

$$\tfrac{1}{2}\,(\varepsilon^{-B} - \varepsilon^{-B\,(b/a)}) = (\varepsilon^{-B\,(t_2/t_1)} - \varepsilon^{-B\,(b/a)\,(t_2/t_1)}) \qquad (63)$$

This equation shows that there corresponds a definite value of t_2/t_1 for any assigned value of b/a. Since the equation is tran-

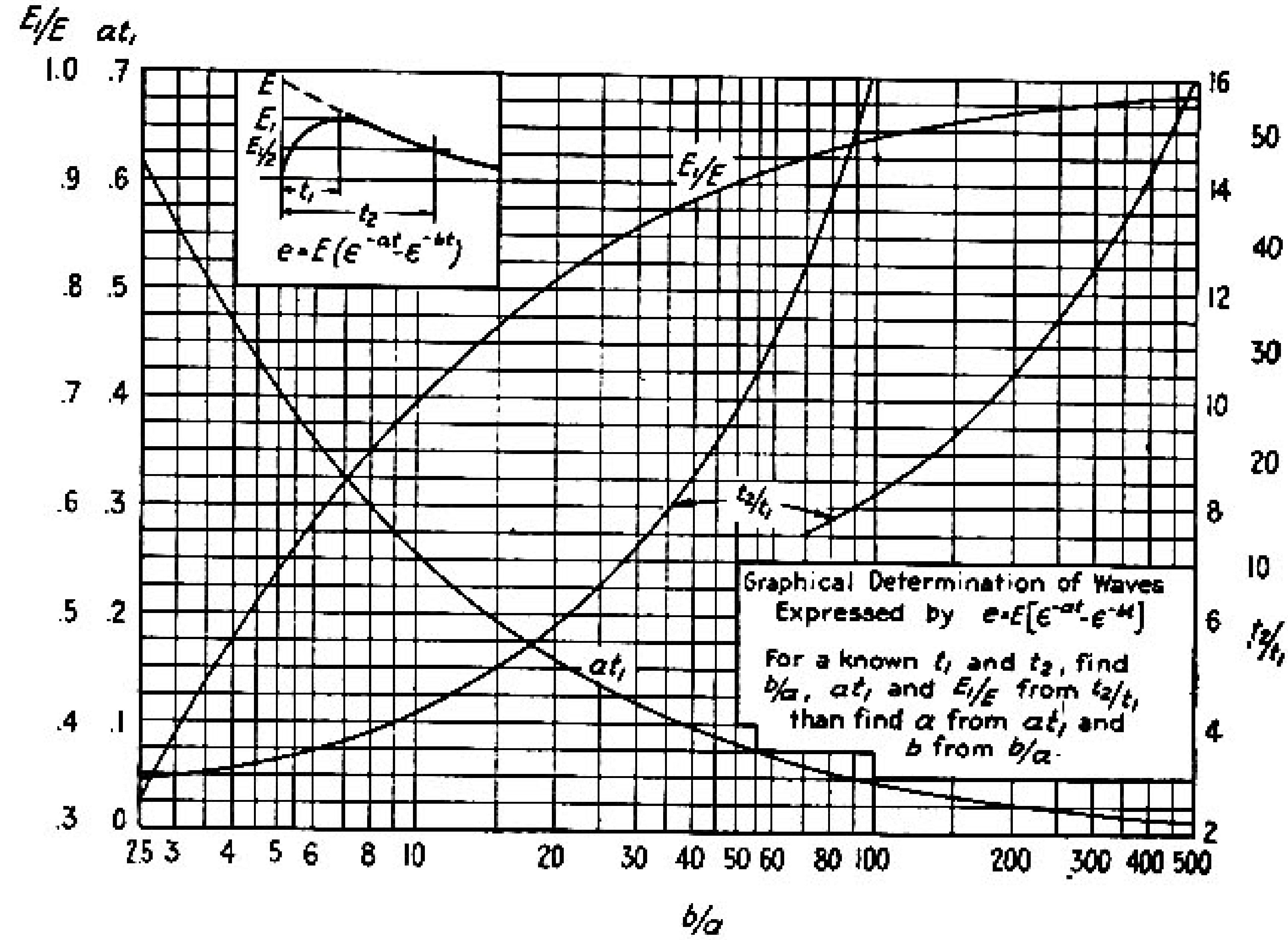


FIG. 7.—Specifications of a Typical Lightning Wave

scendental it is necessary to find t_2/t_1 by plotting or other method of approximation. We now have:

$$\left.\begin{array}{ll} at_1 & \text{as function of } b/a \text{ from Equation (60)} \\ E_1/E & \text{as function of } b/a \text{ from Equation (61)} \\ t_2/t_1 & \text{as function of } b/a \text{ from Equation (63)} \end{array}\right\} \qquad (64)$$

These functions have been plotted * in Fig. 7. As an example of their use let it be required to find parameters a, b, and E to specify a

* "The Solution of Circuits Subjected to Traveling Waves," by H. L. Rorden, *A.I.E.E. Trans.*, 1932.

1000 kv./3.0/21/(+) wave. Then $t_2/t_1 = 7$, and from this value on the t_2/t_1 curve we find $b/a = 28.5$. But for this value of b/a the curves give $at_1 = 0.122$ and $E_1/E = 0.852$. Therefore

$$a = 0.122/t_1 = 0.122/3 = 0.041$$

$$b = 28.5\,a = 28.5 \times 0.041 = 1.15$$

$$E = E_1/0.852 = 1000/0.852 = 1175$$

and the wave is specified as

$$e = 1175\,(\varepsilon^{-0.041t} - \varepsilon^{-1.15t})$$

The foregoing refinements are not ordinarily necessary for determining the parameters of a typical lightning wave, because b is usually very large compared to a, and thus the tail of the wave is practically independent of b, so that (62) may be approximated by

$$e \cong E\varepsilon^{-at} \text{ for } t \gg t_1 \tag{65}$$

Therefore, if t_2 and t_3 are two points well down on the tail of the wave, we have

$$e_2 \cong E\varepsilon^{-at_2}$$

$$e_3 \cong E\varepsilon^{-at_3}$$

and

$$\frac{e_2}{e_3} \cong \varepsilon^{-a\,(t_2 - t_3)}$$

therefore

$$a \cong \frac{\log\,(e_2/e_3)}{t_3 - t_2} \tag{66}$$

and

$$E \cong e_2\varepsilon^{at_2} \cong e_3\varepsilon^{at_3} \tag{67}$$

Then for a point t_0 about half way up on the front of the wave

$$e = e_0 = E\,(\varepsilon^{-at_0} - \varepsilon^{-bt_0})$$

from which

$$\varepsilon^{-bt_0} = (\varepsilon^{-at_0} - e_0/E)$$

or

$$b = \frac{1}{t_0}\log\left(\frac{E}{E\varepsilon^{-at_0} - e_0}\right) \tag{68}$$

SUMMARY OF CHAPTER I

The differential equations of the ideal single-circuit transmission line characterized by the four circuit constants R, L, C, G are:

$$\frac{\partial^2 e}{\partial x^2} = [RG + (RC + GL)\, p + L\, C\, p^2]\, e$$

$$\frac{\partial^2 i}{\partial x^2} = [RG + (RC + GL)\, p + L\, C\, p^2]\, i$$

If $RC = GL$, these equations are solved by traveling-wave functions

$$e = \varepsilon^{-R/L\, t}\, [f\,(x - vt) + F\,(x + vt)] = Z\,(i - i')$$

$$i = \varepsilon^{-R/L\, t}\, [f\,(x - vt) - F\,(x + vt)]\, \sqrt{\frac{C}{L}} = Y\,(e - e')$$

where

$f\,(x - vt)$ = forward wave.

$F\,(x + vt)$ = backward wave.

$$v = \frac{1}{\sqrt{L\, C}} = \text{velocity of propagation.}$$

$$Z = \frac{1}{Y} = \sqrt{\frac{L}{C}} = \frac{e}{i} = -\frac{e'}{i'} = \text{surge impedance.}$$

If the losses are negligible, the exponential decrement factor vanishes. The energy content of a traveling wave is

$$W = \int eidt = Y\int e^2dt = Z\int i^2dt$$

$$= \sqrt{LC}\int eidx = C\int e^2dx = L\int i^2dx$$

This energy resides equally in the electromagnetic and the electrostatic fields.

When a traveling wave impinges on an impedance $Z_0\,(p)$ at the end of the transmission line of surge impedance z upon which it is traveling, it gives rise to a reflected wave

$$e' = \frac{Z_0\,(p) - z}{Z_0\,(p) + z}\, e$$

and the total potential at the transition point is

$$e'' = e + e' = \frac{2\, Z_0\,(p)}{Z_0\,(p) + z}\, e$$

The complete relationships at a transition point are derived in the text.

The shape of a traveling wave may be specified in terms of rectangular components, combinations of exponentials, and compounding of simple waves. For most purposes the typical lightning wave may be represented by the difference of two exponentials

$$e = E\,(\varepsilon^{-at} - \varepsilon^{-bt})$$

A graph is given in Fig. 7 for readily determining the parameters E, a, and b for a wave of any specified crest, front, and tail.

Solutions for waves of arbitrary shape may be found by means of Duhamel's theorem.

CHAPTER II

CALCULATION OF TYPICAL CASES

By way of illustration of the general equations and methods of analysis described in Chapter I, consider the circuit shown in Fig. 8, which is a special case of Fig. 2 where

$$Z_1(p) = pL_1 \qquad z_1 = z_1$$

$$Z_g(p) = 1/pC \qquad z_2 = z_2$$

$$Z_2(p) = R_2 \qquad e_1 = E(\varepsilon^{-at} - \varepsilon^{-bt})$$

It is required to find the reflected and transmitted waves, as well as the voltage at the grounding impedance and across the series impedances, and the corresponding currents.

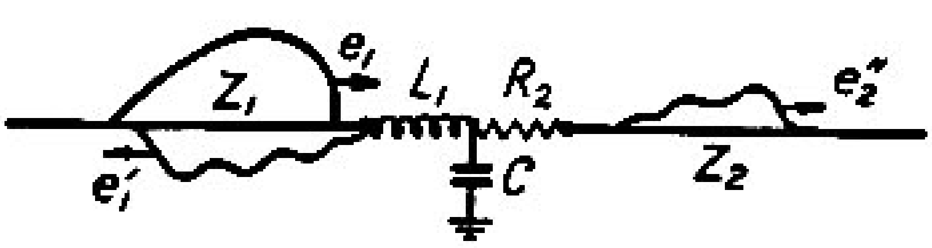


FIG. 8.—Typical Transition Point

By Equation (41) the total impedance at the transition point is

$$Z_0(p) = pL_1 + \cfrac{1}{pC + \cfrac{1}{z_2 + R_2}}$$

$$= \frac{(z_2 + R_2)\,L_1\,C\,p^2 + pL_1 + (z_2 + R_2)}{(z_2 + R_2)\,pC + 1}$$

The reflected voltage wave then is, by (42)

$$e_1' = \left\{\frac{(z_2+R_2)\,L_1C\,p^2 + [L_1 - z_1\,(z_2+R_2)\,C]\,p + (R_2+z_2-z_1)}{(z_2+R_2)\,L_1C\,p^2 + [L_1+z_1\,(z_2+R_2)\,C]\,p + (R_2+z_2+z_1)}\right\}e_1$$

This is the operational equation. The solution is the difference of the solutions for $E\varepsilon^{-at}$ and $E\varepsilon^{-bt}$. Substituting the former, there is

$$e_{1a}' = \frac{p^2 + 2\,\alpha p + A}{p^2 + 2\,\beta p + B} E\varepsilon^{-at}$$

in which

$$2\,\alpha = \left[\frac{1}{(z_2 + R_2)\,C} - \frac{z_1}{L_1}\right]$$

$$2\,\beta = \left[\frac{1}{(z_2 + R_2)\,C} + \frac{z_1}{L_1}\right]$$

$$A = \frac{R_2 + z_2 - z_1}{(z_2 + R_2)\,L_1 C}$$

$$B = \frac{R_2 + z_2 + z_1}{(z_2 + R_2)\,L_1 C}$$

Applying the shifting theorem, Appendix, there is

$$e_{1a}' = E\varepsilon^{-at} \frac{p^2 + 2\,(\alpha - a)\,p + (A - 2\,\alpha a + a^2)}{p^2 + 2\,(\beta - a)\,p + (B - 2\,\beta a + a^2)} 1$$

and making use of Equations (69), (70), and (71) in the Appendix, there results (calling $\omega^2 = B - \beta^2$),

$$e_{1a}' = E\frac{A - 2\,\alpha a + a^2}{B - 2\,\beta a + a^2}\varepsilon^{-at} + E\varepsilon^{-\beta t}\left\{\left[1 - \frac{A - 2\,\alpha a + a^2}{B - 2\,\beta a + a^2}\right]\cos\omega t\right.$$
$$\left. + \left[\frac{2\,(\alpha - a)}{\omega} - \frac{(\beta - a)}{\omega}\left(1 + \frac{A - 2\,\alpha a + a^2}{B - 2\,\beta a + a^2}\right)\right]\sin\omega t\right\}$$

An exactly similar expression results for $E\varepsilon^{-bt}$, and the solution for e_1' therefore is

$$e_1' = e_{1a}' - e_{1b}' = E\left[\frac{A - 2\,\alpha a + a^2}{B - 2\,\beta a + a^2}\varepsilon^{-at} - \frac{A - 2\,\alpha b + b^2}{B - 2\,\beta b + b^2}\varepsilon^{-bt}\right]$$
$$+ E\varepsilon^{-\beta t}\left\{\left[\frac{A - 2\,\alpha b + b^2}{B - 2\,\beta b + b^2} - \frac{A - 2\,\alpha a + a^2}{B - 2\,\beta a + a^2}\right]\cos\omega t\right.$$
$$+ \left[\frac{b - a}{\omega} - \frac{(\beta - a)}{\omega}\frac{A - 2\,\alpha a + a^2}{B - 2\,\beta a + a^2}\right.$$
$$\left.\left. + \frac{(\beta - b)}{\omega}\frac{A - 2\,\alpha b + b^2}{B - 2\,\beta b + b^2}\right]\sin\omega t\right\}$$

The current in the reflected wave is

$$i_1' = -\frac{e_1'}{z_1}.$$

and the total current flowing into Z_0 is the sum of the incident and reflected currents, or

$$i_0 = i_1 + i_1' = \frac{e_1 - e_1'}{z_1}$$

This current causes a voltage drop across L_1 of

$$E_1 = p\,L_1\,i_0 = \frac{L_1}{z_1}\,p\,(e_1 - e_1')$$

Now the total voltage at the transition point is the sum of the incident and reflected waves, or

$$e_0 = e_1 + e_1'$$

and therefore the voltage across the capacitor C is

$$e = e_0 - E_1 = (e_1 + e_1') - \frac{L_1}{z_1}\,p\,(e_1 - e_1')$$

The current which will flow into the capacitor C due to this voltage is

$$i_g = C\,pe = Cp\,(e_1 + e_1') - \frac{CL_1}{z_1}\,p^2\,(e_1 - e_1')$$

Subtracting the current i_g which flows to ground from the total current i_0 flowing through L_1 gives the current

$$i_2'' = i_0 - i_g = (e_1 - e_1')\frac{1}{z_1} - C\,p\,(e_1 + e_1') + \frac{CL_1}{z_1}\,p^2\,(e_1 - e_1')$$

$$= (1 + CL_1p^2)\,\frac{e_1 - e_1'}{z_1} - C\,p\,(e_1 + e_1')$$

which flows through R_2 and enters line z_2. This current causes a drop

$$E_2 = R_2 i_2'' = \frac{R_2}{z_1}\,(1 + CL_1p^2)\,(e_1 - e_1') - R_2C\,p\,(e_1 + e_1')$$

in the resistor, so that the transmitted voltage wave is

$$e_2'' = e - R_2 i_2''$$

$$= (1 + R_2Cp)\,(e_1 + e_1') - \frac{1}{z_1}\,(R_2CL_1p^2 + L_1p + R_2)\,(e_1 - e_1')$$

and its accompanying current wave is $i_2'' = e_2''/z_2$. Of course, if i_2'' has already been calculated, then $e_2'' = z_2 i_2''$; but more likely, in any actual case, the voltages e_1', e_0, e, and e_2'' will be calculated first, and then the other quantities are readily found: $i_1' = -e_1'/z_1$, $e_0 = e_1' + e_1$, $E_1 = e_0 - e$, $E_2 = e - e_2''$, $i_0 = i_1 + i_1'$, $i_2'' = e_2''/z_2$, $i_g = i_0 - i_2''$. It is evident that the order in which the unknowns are found is immaterial, and the procedure can be varied to suit the convenience of the calculator. If in a particular instance it is only desired to find the transmitted voltage wave, and the other voltages and currents at the transition point are of no immediate concern, it would be foolish to go through all the calculations outlined above. In such a case the refraction operator given in Chapter I as Equation (48) would be used.

It is sometimes possible to obtain the solution for a given transition point by an ingenious interpretation of the solution for an entirely

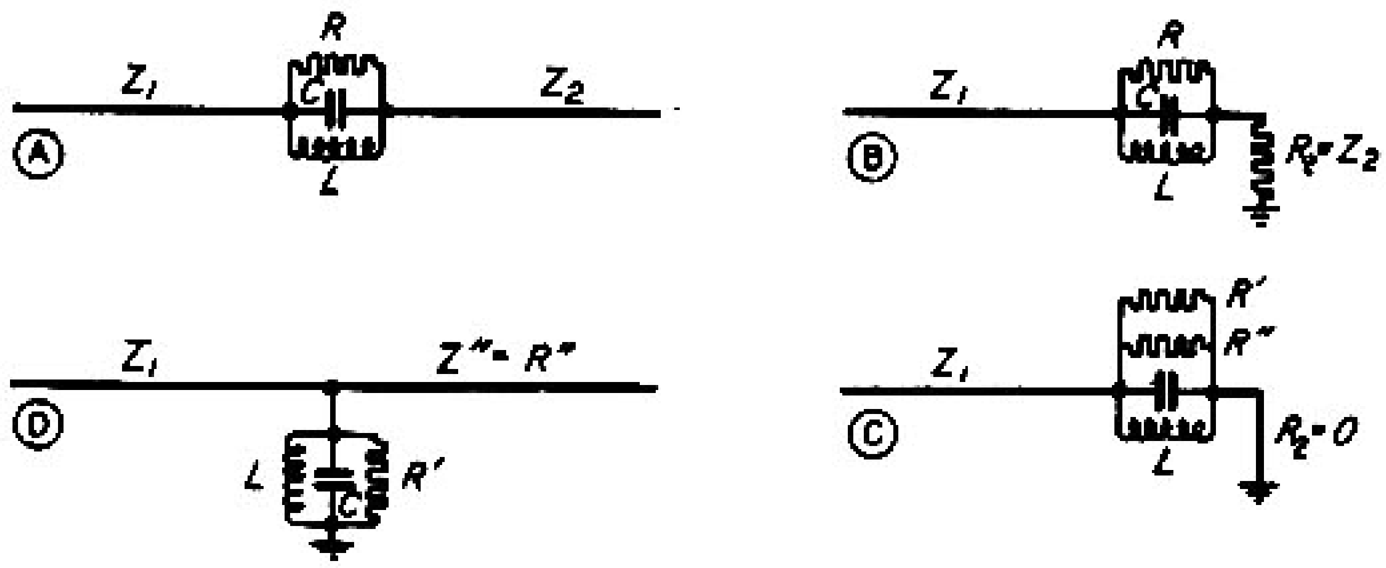


FIG. 9.—Mutually Convertible Networks

different case. This possibility is illustrated in Fig. 9, in which the variations shown in (*a*), (*b*), (*c*), and (*d*) are mutually convertible combinations, and therefore, under the conditions of a proper interpretation, the same equations can be made to apply to each. Moreover, by letting the different constants become infinite or vanish as required, this single set-up contains 24 special cases.

Since the surge impedance of a connected line, as Z_2, enters in the generalized impedance $Z_0(p)$ in exactly the same way as a resistance to ground R_2, it is evident that Fig. 9*B* is the analytic equivalent of Fig. 9*A*, so that the same equations apply to each when R_2 and Z_2 are exchanged. In Fig. 9*C* the grounding resistance $R_2 = 0$, and R has been expressed as the resultant of two resistances R' and R'' in parallel. But now again, the form of the generalized impedance is not altered if R'' be replaced by the surge impedance $Z'' = R''$ of an outgoing line. Fig. 9*D* then follows as a natural consequence.

Although such an equivalence is quite interesting and allows many combinations to be expressed by a single general equation, yet under many conditions it is quicker, and there is less chance for error, if the result is derived directly from the differential equations of the particular case. In performing a reduction from the general equation

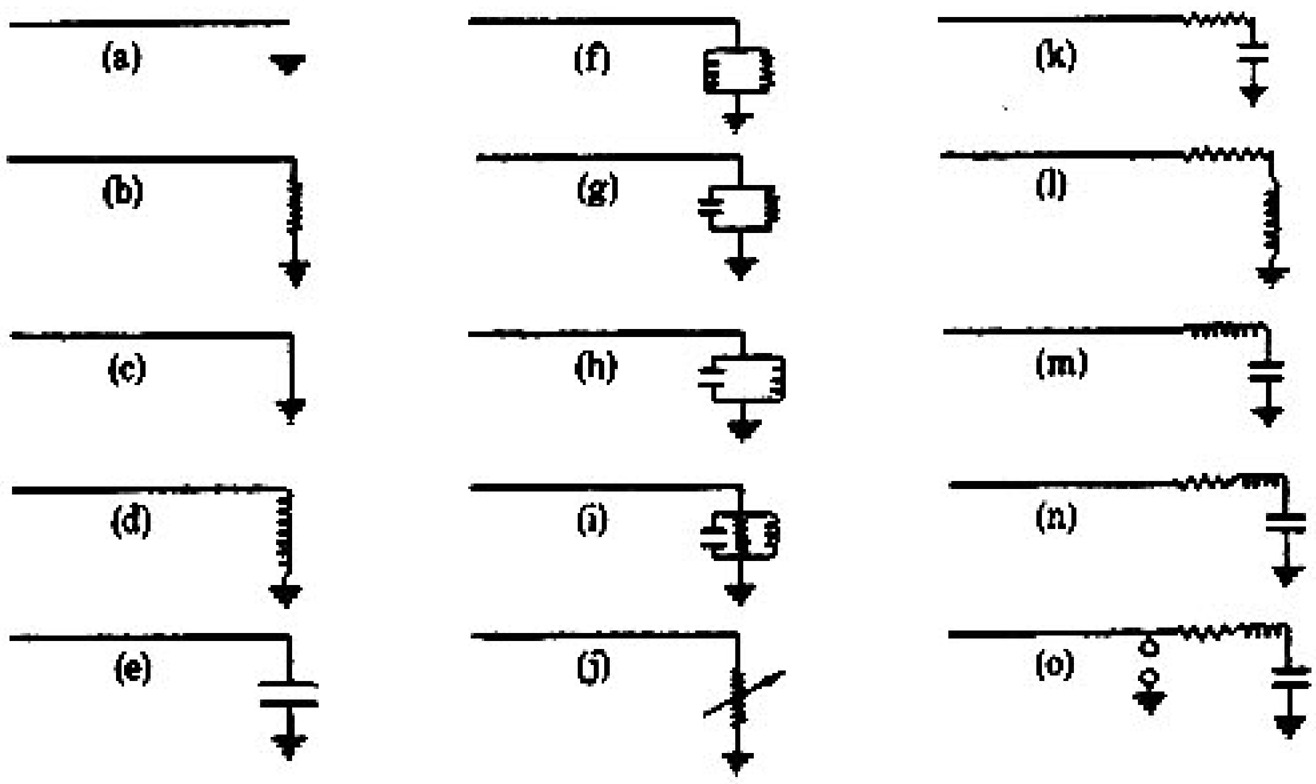


FIG. 10.—Terminal Conditions on Single-Conductor Circuits

applying to Fig. 9, great care must be taken in evaluating the indeterminates which appear for limiting values (zero or infinity) of the constants, and in transforming functions of imaginary variables to functions of real variables, etc.

Most of the circuits encountered in transmission systems reduce to relatively simple combinations under traveling-wave conditions. Thus a current-limiting reactor is simply an inductance, a transformer acts as a capacitance and inductance in parallel, rotating machines are practically the same as short transmission lines, etc. The circuits shown in Figs. 10, 11, and 12 are representative of those which simulate many actual conditions. By a fortunate coincidence, nearly all these cases are included by two general equations, through an appropriate adjustment of the coefficients thereof. The process of deriving

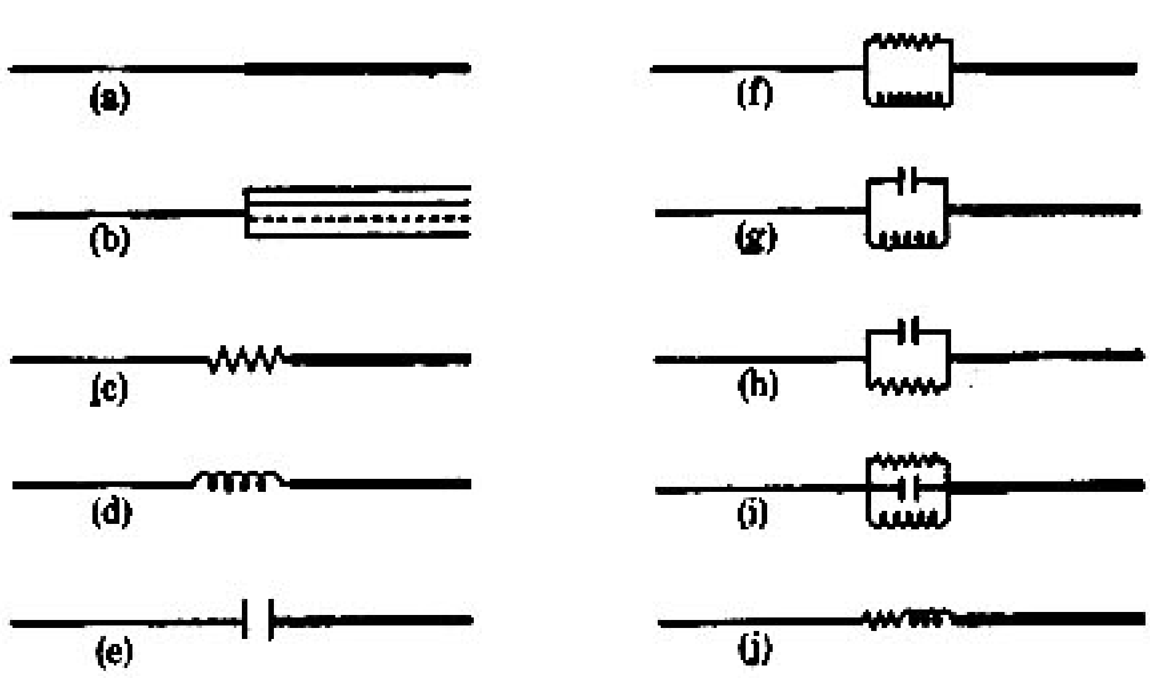


FIG. 11.—Junctions between Single-Conductor Circuits

these equations for any of these particular cases is in no wise different from that of Fig. 8. Therefore all that is necessary here is to give the equations and tables with the proper coefficients for each case. Working out some of the cases will prove profitable to the student

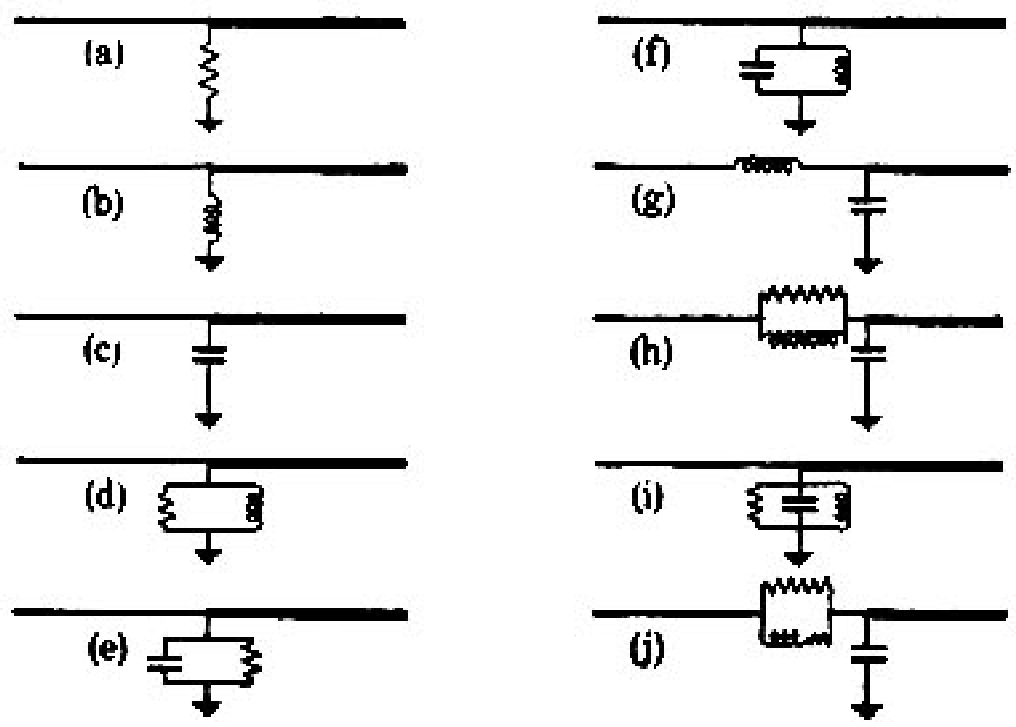


FIG. 12.—Junctions between Single-Conductor Circuits

approaching the subject of traveling waves for the first time. The two equations for an exponential applied wave

$$e = E\varepsilon^{-at}$$

are

$$EA\left[\frac{a+\alpha}{a-\beta}\varepsilon^{-at} - \frac{\alpha+\beta}{a-\beta}\varepsilon^{-\beta t}\right] \qquad \text{(I)}$$

$$EA\left[\frac{\omega_0^2 - 2a\alpha + a^2}{\omega_0^2 - 2a\beta + a^2}\varepsilon^{-at} - \frac{2(\alpha-\beta)\varepsilon^{-\beta t}}{\omega(\omega_0^2 - 2a\beta + a^2)}\right.$$

$$\left.\left\{(\omega_0^2 - a\beta)\sin\omega t + a\omega\cos\omega t\right\}\right] \qquad \text{(II)}$$

in which

$$\omega_0^2 = 1/LC \quad \text{and} \quad \omega^2 = \omega_0^2 - \beta^2$$

Tables I, II, and III give the reflected waves e' and the transmitted waves e''. In all cases the total voltage at the transition point is $e_0 = e + e'$. The reflected and transmitted current waves are $i' = -e'/z_1$ and $i'' = e''/z_2$ respectively, and the total current at the transition point is $i_0 = i + i'$.

Figs. 13 to 24 inclusive illustrate in a general way the variation in the shape of the reflected and transmitted waves caused by different circuit conditions at the transition point. Each figure shows a sketch of the circuit under consideration, defines the variable parameters, and indicates the range in shape of the reflected and transmitted waves corresponding to both an infinite rectangular incident wave and a characteristic 20-ms. incident wave. Although these cases are fairly self-explanatory, a few pertinent remarks may be helpful.

Fig. 13. Line Closed by a Resistance R_0.—This figure shows a transmission line of surge impedance Z grounded through various values of terminal resistances R_0. When $R_0 = 0$ the reflected voltage wave e' is negative and equal in magnitude to the incident wave e, so that the total voltage at the transition point is $e_0 = 0$. On the other hand, the current reflection is positive and equal to i, so that the total current which flows into the ground connection is $i_0 = i + i' = 2\,i$.

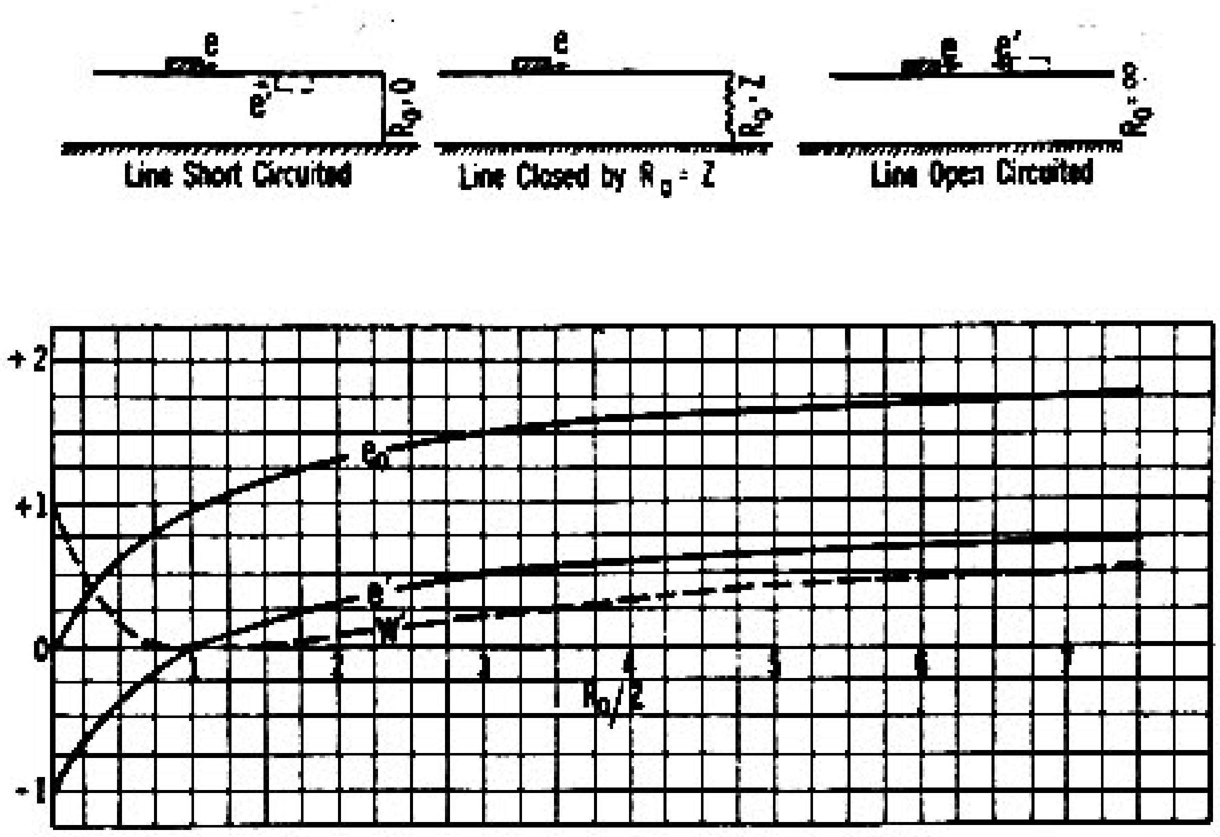


FIG. 13.—Line Closed by a Resistance R_0

$$e' = \left(\frac{R_0 - Z}{R_0 + Z}\right) e, \quad e_0 = e + e' = \left(\frac{2\,R_0}{R_0 + Z}\right) e$$

For the critical value of resistance $R_0 = Z$, that is a resistance equal to the surge impedance of the transmission line, there is neither voltage nor current reflection and the entire energy of the incident wave is absorbed in the resistor. But as R_0 is increased beyond this critical value of $R_0 = Z$, the voltage reflection turns positive, and eventually, for $R_0 = \infty$ —an open circuit—the voltage reflection is positive and equal in value to the incident wave, so that the total voltage at the transition point is double that of the incident wave, $e_0 = e + e' = 2\,e$. Thus a lightning surge striking the open end of a transmission line will double in value and may then spark over the insulation. The current reflection for an open-end line has full negative value, and the total current is therefore zero. The way in

TABLE I

Fig.	Equation	α	β	A
10-a	$e' = e$			
10-b	$e' = \frac{R - Z}{R + Z} e$			
10-c	$e' = - e$			
10-d	$e' =$ (I.)	$\frac{Z}{L}$	$\frac{Z}{L}$	1
10-e	$e' =$ (I.)	$\frac{1}{C Z}$	$\frac{1}{C Z}$	-1
10-f	$e' =$ (I.)	$\frac{Z R}{L (R - Z)}$	$\frac{Z R}{L (R + Z)}$	$\frac{R - Z}{R + Z}$
10-g	$e' =$ (I.)	$\frac{R - Z}{Z R C}$	$\frac{R + Z}{Z R C}$	-1
10-h	$e' =$ (II.)	$-\frac{1}{2 C Z}$	$\frac{1}{2 C Z}$	-1
10-i	$e' =$ (II.)	$\frac{Z - R}{2 R C Z}$	$\frac{Z + R}{2 R C Z}$	-1

10-j	$e' = e - z\, i_0$			
10-k	$e' =$ (I.)	$\dfrac{1}{C\,(Z - R)}$	$\dfrac{1}{C\,(Z + R)}$	$\dfrac{R - Z}{R + Z}$
10-l	$e' =$ (I.)	$\dfrac{Z - R}{L}$	$\dfrac{Z + R}{L}$	1
10-m	$e' =$ (II.)	$\dfrac{-Z}{2\,L}$	$\dfrac{Z}{2\,L}$	1
10-n	$e' =$ (II.)	$\dfrac{R - Z}{2\,L}$	$\dfrac{R + Z}{2\,L}$	1
10-o	Same as 10-n before gap sparkover Same as 10-c after sparkover			

TABLE II

Fig.	Equation	α	β	A
11-a	$e' = \frac{Z_2 - Z_1}{Z_2 + Z_1} e$ $e'' = \frac{2 Z_2}{Z_2 + Z_1} e$			
11-b	$e' = \frac{1 - Z_1 Y_0}{1 + Z_1 Y_0} e$ $e'' = \frac{2}{1 + Z_1 Y_0} e$	Y_0 = total admittance of all outgoing lines in parallel		
11-c	$e' = \frac{Z_2 - Z_1 + R}{Z_2 + Z_1 + R} e$ $e'' = \frac{2 Z_2}{Z_2 + Z_1 + R} e$			
11-d	$e' = (\mathrm{I.})$ $e'' = \frac{\alpha - \beta}{\alpha - \beta} (\epsilon^{-\alpha t} - \epsilon^{-\beta t})$	$\frac{Z_1 - Z_2}{L}$ $\frac{Z_1 - Z_2}{L}$	$\frac{Z_1 + Z_2}{L}$ $\frac{Z_1 + Z_2}{L}$	1
11-e	$e' = (\mathrm{I.})$ $e'' = (\mathrm{I.})$	$\frac{1}{C(Z_1 - Z_2)}$ 0	$\frac{1}{C(Z_1 + Z_2)}$ $\frac{1}{C(Z_1 + Z_2)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$ $\frac{2 Z_2}{Z_2 + Z_1}$

11-f	$e' = $ (I.)	$\frac{R(Z_1 - Z_2)}{L(R - Z_1 + Z_2)}$	$\frac{R(Z_1 + Z_2)}{L(R + Z_1 + Z_2)}$	$\frac{R + Z_2 - Z_1}{R + Z_2 + Z_1}$
	$e'' = $ (I.)	$-\frac{R}{L}\frac{Z_2}{(R + Z_2)}$	$\frac{R(Z_1 + Z_2)}{L(R + Z_1 + Z_2)}$	$\frac{2(Z_2 + R)}{R + Z_1 + Z_2}$
11-g	$e' = $ (II.)	$\frac{1}{2C(Z_2 - Z_1)}$	$\frac{1}{2C(Z_2 + Z_1)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
	$e'' = $ (II.)	0	$\frac{1}{2C(Z_2 + Z_1)}$	$\frac{2Z_2}{Z_2 + Z_1}$
11-h	$e' = $ (I.)	$\frac{R_1 - Z_1 + Z_2}{RC(Z_1 - Z_2)}$	$\frac{R + Z_1 + Z_2}{RC(Z_1 + Z_2)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
	$e'' = $ (I.)	$-\frac{1}{RC}$	$\frac{R + Z_1 + Z_2}{RC(Z_1 + Z_2)}$	$\frac{2Z_2}{Z_2 + Z_1}$
11-i	$e' = $ (II.)	$\frac{Z_2 - Z_1 + R}{2RC(Z_2 - Z_1)}$	$\frac{Z_2 + Z_1 + R}{2RC(Z_2 + Z_1)}$	$\frac{Z_2 - Z_1}{Z_2 + Z_1}$
	$e'' = $ (II.)	$\frac{1}{2RC}$	$\frac{Z_2 + Z_1 + R}{2RC(Z_2 + Z_1)}$	$\frac{2Z_2}{Z_2 + Z_1}$
11-j	$e' = $ (I.)	$\frac{Z_1 - Z_2 - R}{L}$	$\frac{Z_1 + Z_2 + R}{L}$	1
	$e'' = \frac{-2Z_2}{\alpha - \beta}(\epsilon^{\alpha t} - \epsilon^{-\beta t})$		$\frac{Z_1 + Z_2 + R}{L}$	

TABLE III

Fig.	Equation	α	β	A
12-a	$e' = \dfrac{Z_2 R - Z_1 R - Z_1 Z_2}{Z_2 R + Z_1 R + Z_1 Z_2}$ $e'' = \dfrac{2 Z_2 R}{Z_2 R + Z_1 R + Z_1 Z_2}$			
12-b	$e' = $ (I.) $e'' = \dfrac{\alpha + \beta}{a - \beta} \dfrac{E}{\alpha} (a \epsilon^{-at} - \beta \epsilon^{-\beta t})$	$\dfrac{Z_1 Z_2}{L (Z_2 - Z_1)}$	$\dfrac{Z_1 Z_2}{L (Z_2 + Z_1)}$	$\dfrac{Z_2 - Z_1}{Z_2 + Z_1}$
12-c	$e' = $ (I.) $e'' = E \dfrac{\alpha + \beta}{a - \beta} (\epsilon^{-at} - \epsilon^{-\beta t})$	$\dfrac{Z_2 - Z_1}{Z_1 Z_2 C}$ $\dfrac{Z_2 - Z_1}{Z_1 Z_2 C}$	$\dfrac{Z_2 + Z_1}{Z_1 Z_2 C}$ $\dfrac{Z_2 + Z_1}{Z_1 Z_2 C}$	-1
12-d	$e' = $ (I.) $e'' = e + e'$	$\dfrac{R Z_1 Z_2}{L (R Z_2 - R Z_1 - Z_1 Z_2)}$	$\dfrac{R Z_1 Z_2}{L (R Z_2 + R Z_1 + Z_1 Z_2)}$	β/α

12-e	$e' = (\text{I.})$ $e'' = -E\dfrac{\alpha+\beta}{\alpha-\beta}(\epsilon^{-\alpha t} - \epsilon^{-\beta t})$	$\dfrac{RZ_2 - RZ_1 - Z_1Z_2}{Z_1Z_2RC}$	$\dfrac{RZ_2 + RZ_1 + Z_1Z_2}{Z_1Z_2RC}$	-1
12-f	$e' = (\text{II.})$ $e'' = e + e'$	$\dfrac{Z_1 - Z_2}{2Z_1Z_2C}$	$\dfrac{Z_1 + Z_2}{2Z_1Z_2C}$	-1
12-i	$e' = (\text{II.})$ $e'' = e + e'$	$\dfrac{Z_1R - Z_2R + Z_1Z_2}{2Z_1Z_2RC}$	$\dfrac{Z_1R + Z_2R + Z_1Z_2}{2Z_1Z_2RC}$	-1
12-g	$e' = (\text{II.})$ $e'' = (\text{II.})$	See Chapter V	for Solutions	
12-h	$e' = (\text{II.})$ $e'' = (\text{II.})$	See Chapter V	for Solutions	
12-j	$e' = (\text{II.})$ $e'' = (\text{II.})$	See Chapter V	for Solutions	

which the reflected voltage wave e', total voltage e_0, and energy content W' in the reflected wave change with R_0/Z is clearly shown in Fig. 13.

Fig. 14. Line Closed by an Inductance L_0.—At the first instant of impact of the incident wave the current through the inductance is zero, and therefore the line acts as an open circuit at that first instant, and the voltage is doubled if the wave front is abrupt. But the inductance

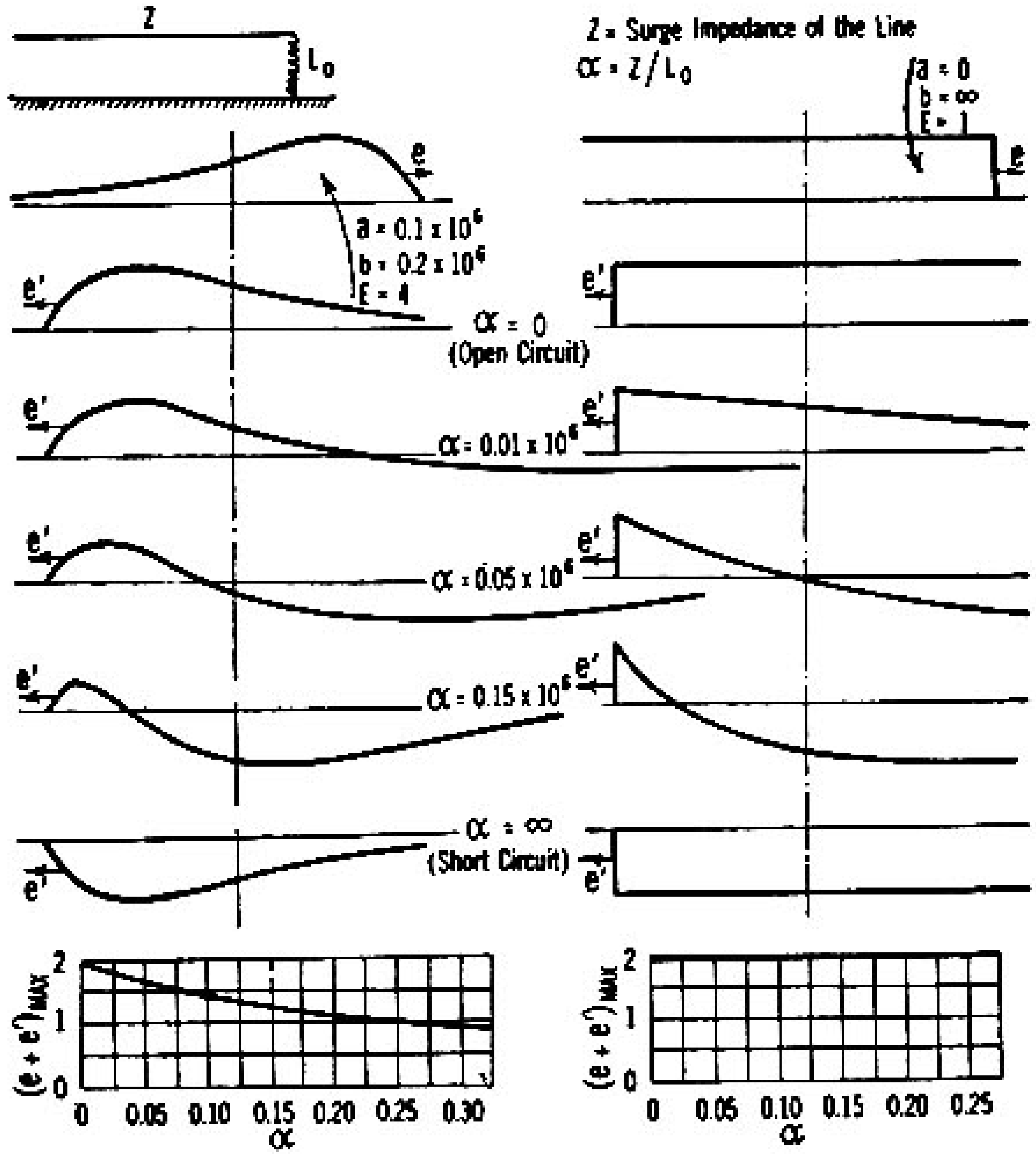


FIG. 14.—Line Closed by an Inductance L_0

$$e = E\left(\epsilon^{-at} - \epsilon^{-bt}\right)$$

$$e' = E\left[\frac{a+\alpha}{a-\alpha}\epsilon^{-at} - \frac{b+\alpha}{b-\alpha}\epsilon^{-bt} + \frac{2\,\alpha\,(a-b)}{(a-\alpha)\,(b-\alpha)}\epsilon^{-\alpha t}\right]$$

gradually permits the passage of current, until it eventually acts as a short circuit, and then the current reflection is equal to that of the incident current wave. As L_0 passes through the intermediate values between zero and infinity, the reflected waves are first elongated and then contracted, and each reflected wave changes sign from positive to negative as it develops. Except for an infinite value of L_0 the total voltage e_0 at the transition point is never double for a wave of finite front.

Fig. 15. Line Closed by a Capacitance C_0.—The form of the equation for the reflected wave is the same as for the case of the inductance given above, but the signs are reversed. Thus the capacitance acts as a short circuit at the first instant, but passes through a transition stage to its final fully charged condition, when it acts as an open circuit.

Fig. 16. Line Closed by R_0 and L_0 in parallel.—Although it is impossible by this combination to dissipate the incident wave com-

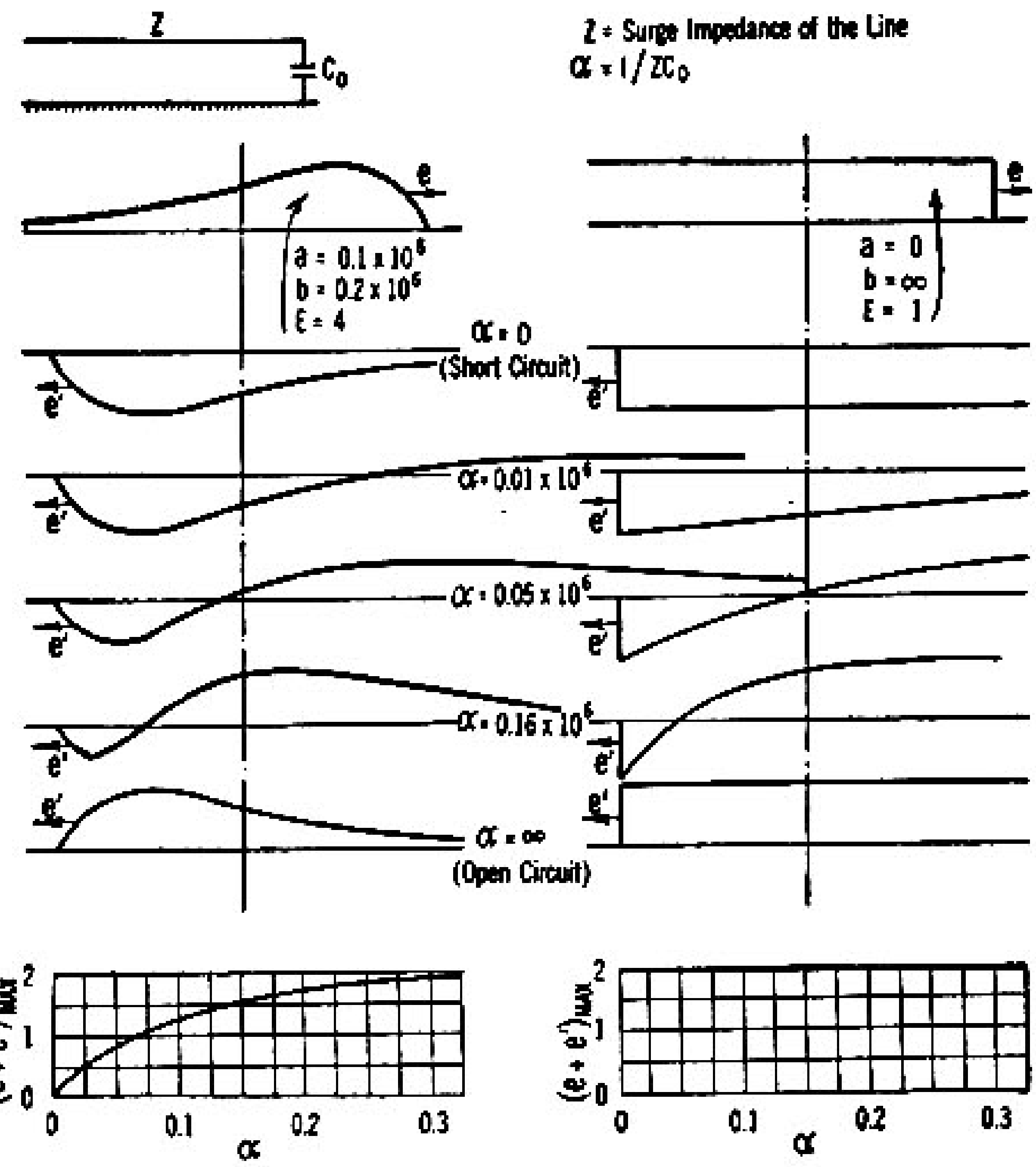


FIG. 15.—Line Closed by a Capacitance C_0

$$e = E\left(\epsilon^{-at} - \epsilon^{-bt}\right)$$

$$e' = -E\left[\frac{a+\alpha}{a-\alpha}\epsilon^{-at} - \frac{b+\alpha}{b-\alpha}\epsilon^{-bt} + \frac{2\alpha(a-b)}{(a-\alpha)(b-\alpha)}\epsilon^{-\alpha t}\right]$$

pletely, yet, for values of R_0 of the order of the surge impedance Z, the reflected wave is considerably reduced in amplitude, and spread out. If R_0 is greater than Z, then the reflection will change sign; but if R_0 is less than Z, the reflection will always be negative.

Fig. 17. Line Closed by R_0 and C_0 in Parallel.—The reflections have the same characteristics as for Fig. 15 when $R_0 > Z$, but if $R_0 < Z$ the reflections can not change sign, but will always be negative.

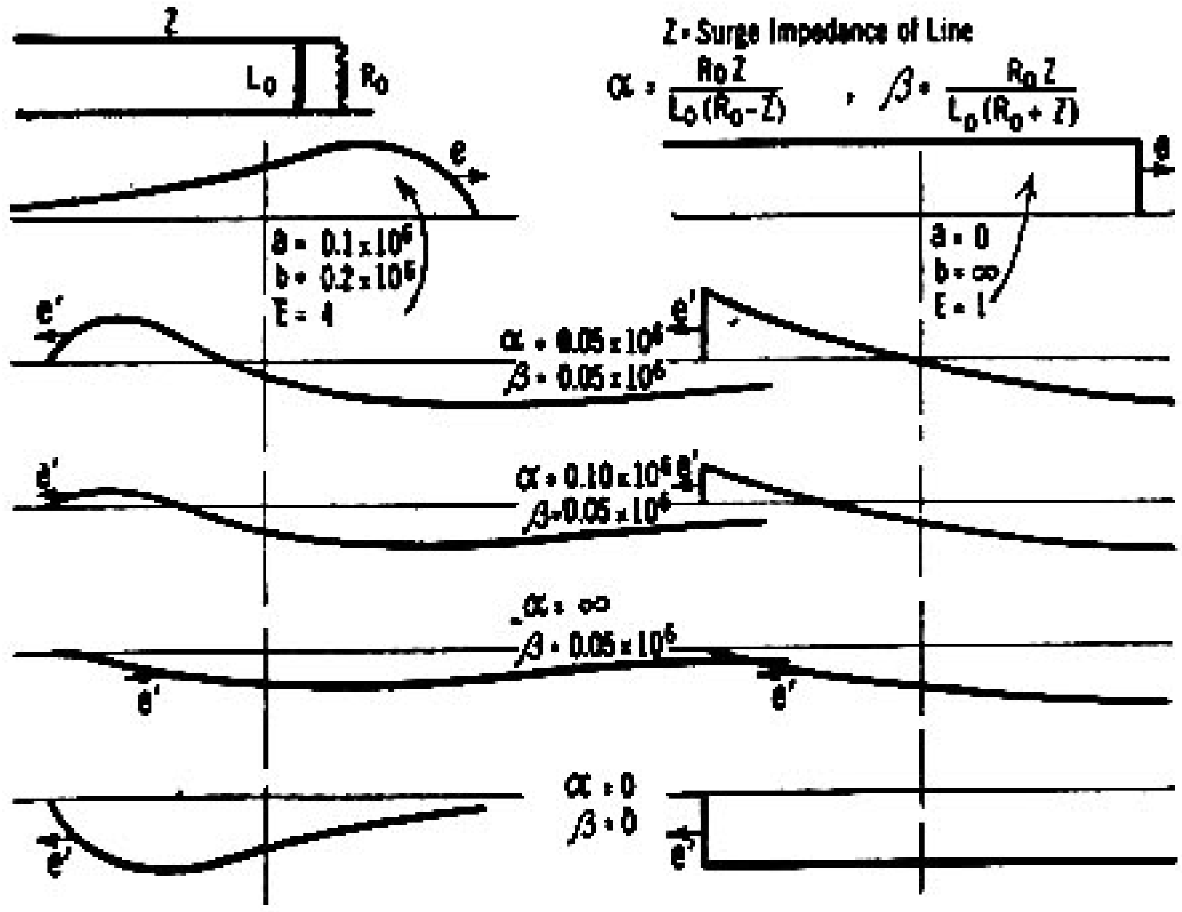


FIG. 16.—Line Closed by R_0 and L_0 in Parallel

$$e = E\left(\epsilon^{-at} - \epsilon^{-bt}\right)$$

$$e' = E\frac{\beta}{\alpha}\left[\frac{a+\alpha}{a-\beta}\epsilon^{-at} - \frac{b+\alpha}{b-\beta}\epsilon^{-bt} + \frac{(\alpha+\beta)(a-b)}{(a-\beta)(b-\beta)}\epsilon^{-\beta t}\right]$$

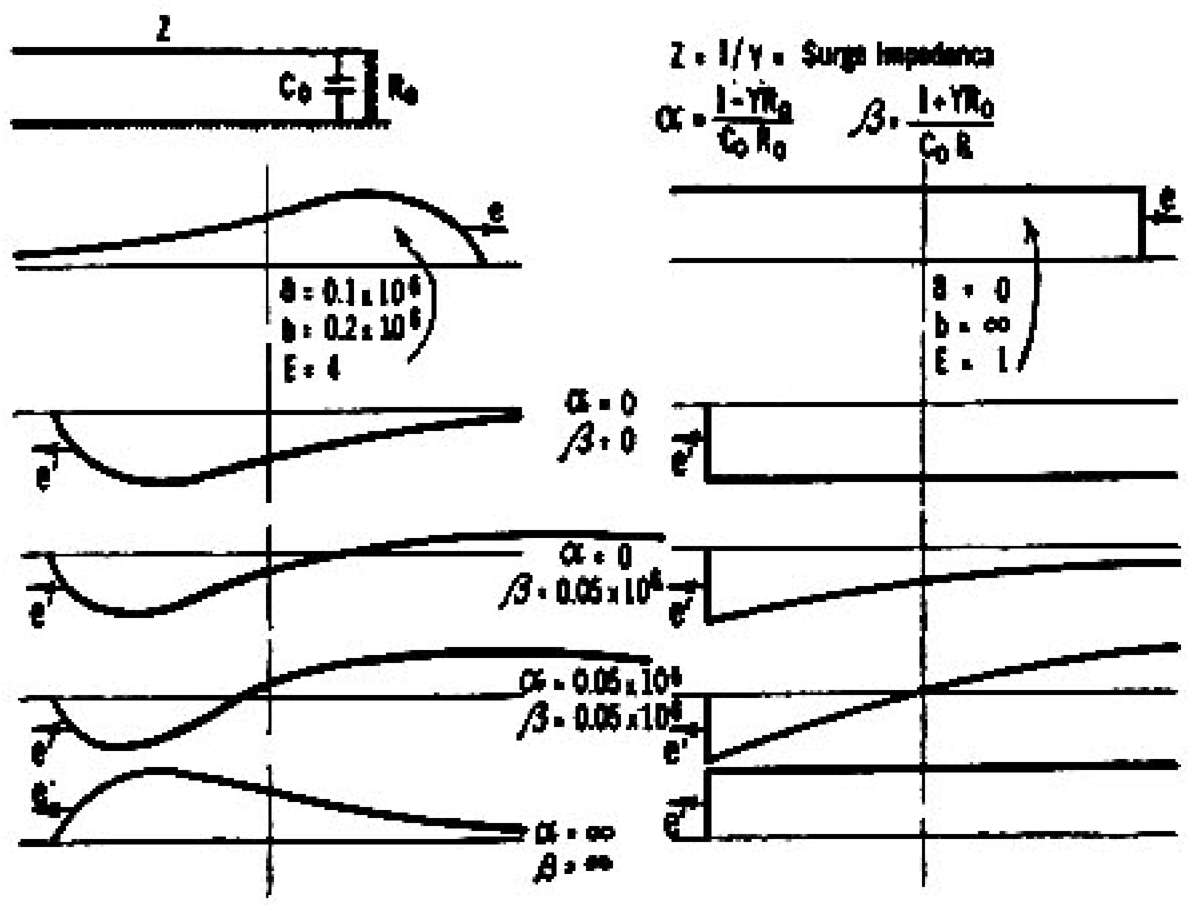


FIG. 17.—Line Closed by R_0 and C_0 in Parallel

$$e = E\left(\epsilon^{-at} - \epsilon^{-bt}\right)$$

$$e' = E\left[-\frac{a-\alpha}{a-\beta}\epsilon^{-at} + \frac{b-\alpha}{b-\beta}\epsilon^{-bt} + \frac{(\alpha-\beta)(a-b)}{(a-\beta)(b-\beta)}\epsilon^{-\beta t}\right]$$

Fig. 18. Line Closed by L_0 and C_0 in Parallel.—Oscillations are shown in the reflected wave, but it does not follow that there will always be an oscillation in such a circuit. In fact, if

$$Z \leq \frac{1}{2}\sqrt{\frac{L}{C}}$$

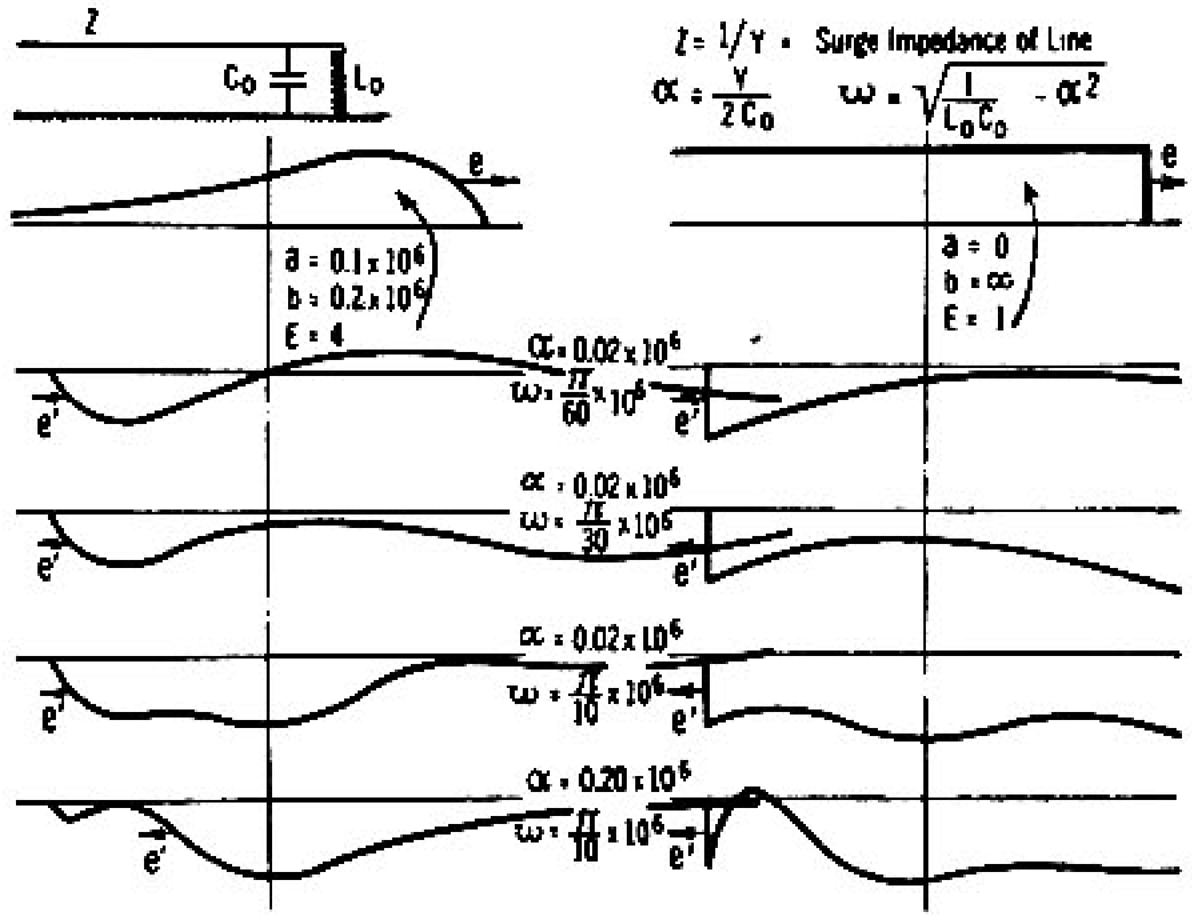


FIG. 18.—Line Closed by L_0 and C_0 in Parallel

$$e = E\,(\epsilon^{-at} - \epsilon^{-bt})$$

$$e' = E\Bigg\{ - \frac{(a+\alpha)^2 + \omega^2}{(a-\alpha)^2 + \omega^2}\,\epsilon^{-at} + \frac{(b+\alpha)^2 + \omega^2}{(b-\alpha)^2 + \omega^2}\,\epsilon^{-bt} + 4\,\frac{\alpha}{\omega}\,\epsilon^{-\alpha t}$$

$$\Bigg[\left(\frac{b\,(b-\alpha)}{(b-\alpha)^2 + \omega^2} - \frac{a\,(a-\alpha)}{(\alpha-\alpha)^2 + \omega^2}\right)\sin\omega t$$

$$-\left(\frac{b\,\omega}{(b-\alpha)^2 + \omega^2} - \frac{a\,\omega}{(a-\alpha)^2 + \omega^2}\right)\cos\omega t\Bigg]\Bigg\}$$

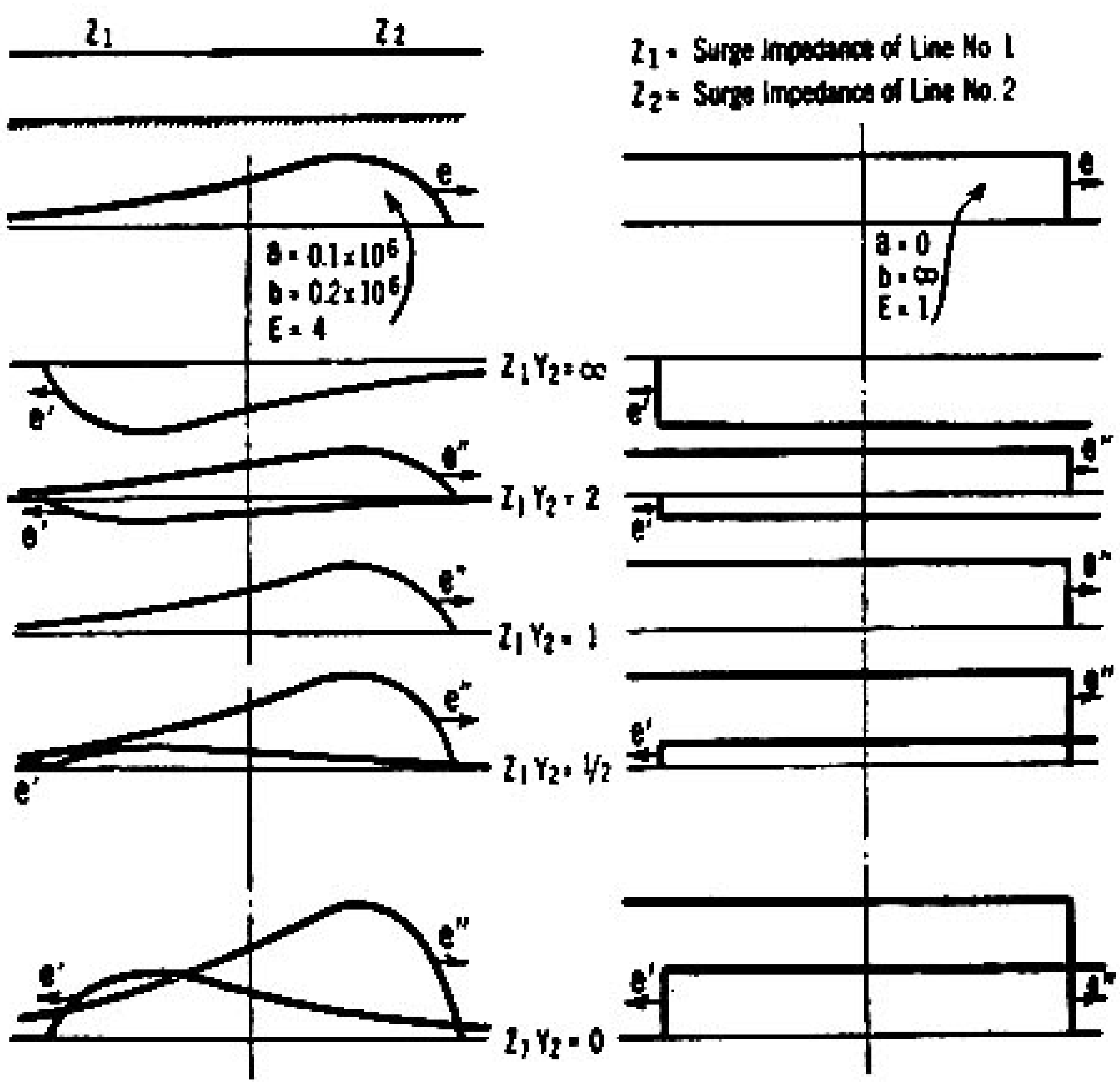


FIG. 19.—Junction of Two Lines

$$e' = \frac{1 - Z_1\,Y_2}{1 + Z_1\,Y_2}\,e, \quad e'' = \frac{2}{1 + Z_1\,Y_2}\,e$$

there will be no oscillations, and the solution degenerates to that of the non-oscillatory case in much the same fashion as in the ordinary well-

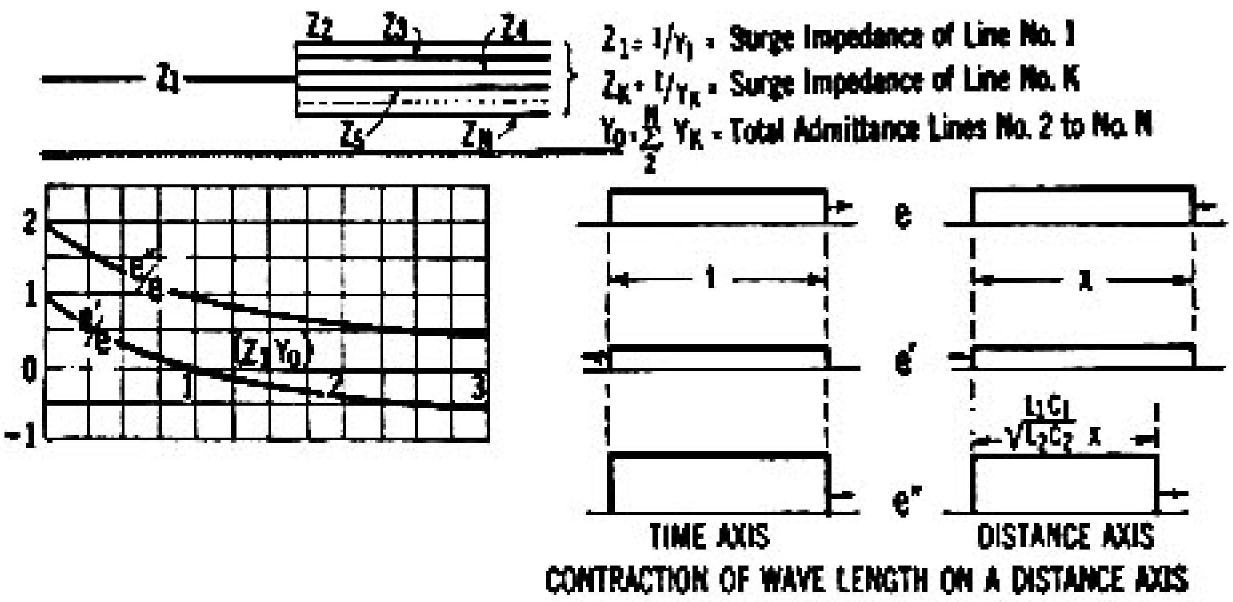


FIG. 20.—Junction of N Lines

$$e = f(x + v_1 t)$$

$$e' = \left(\frac{1 - Z_1 Y_0}{1 + Z_1 Y_0}\right) f(x - v_1 t)$$

$$e'' = \left(\frac{2}{1 + Z_1 Y_0}\right) f(x + v_k t)$$

known L, C, R series circuit. It will be noticed from Fig. 18 that a circuit of this type may retard the development of the wave front for

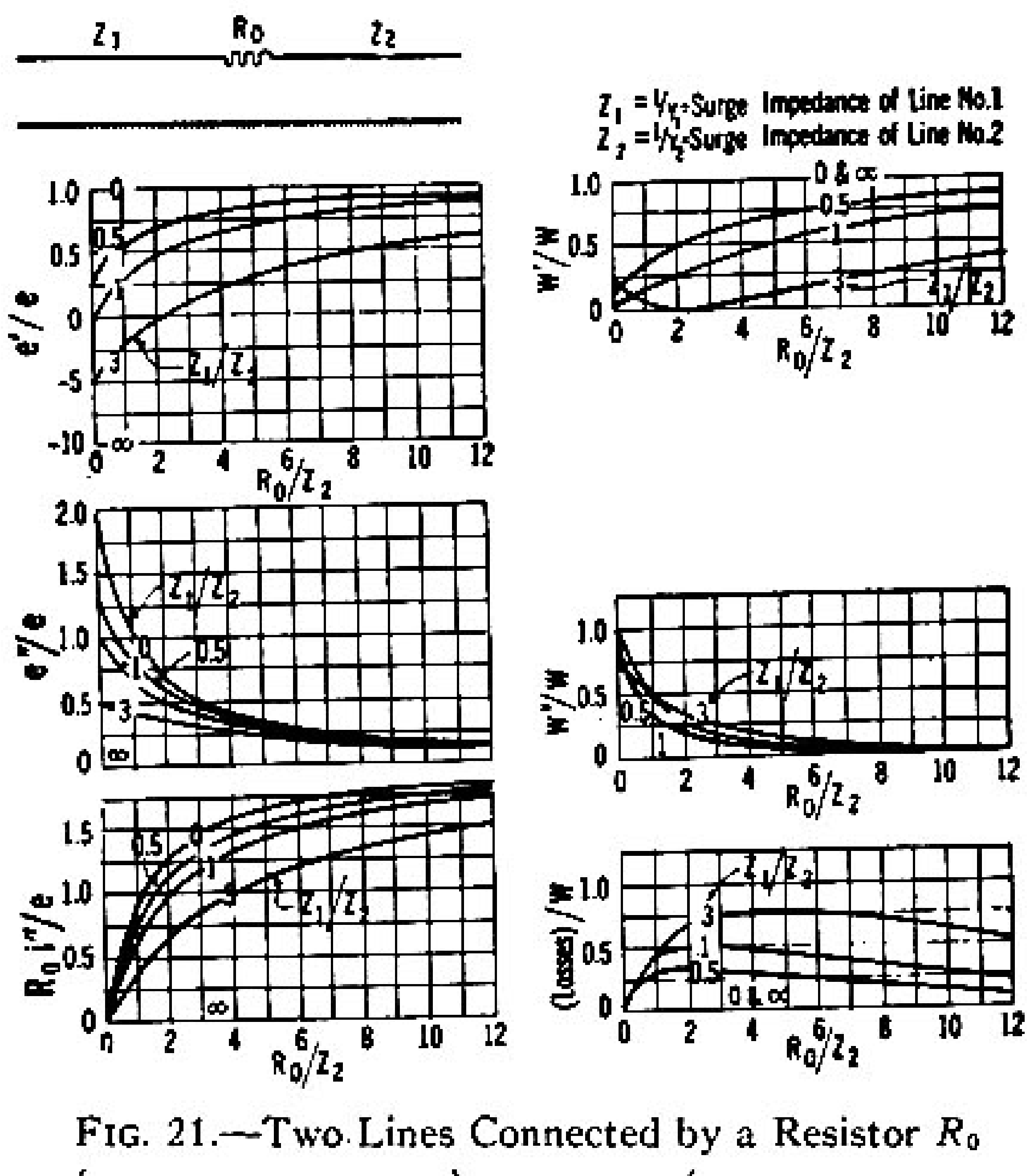


FIG. 21.—Two Lines Connected by a Resistor R_0

$$e' = \left(\frac{1 + R_0/Z_2 - Z_1/Z_2}{1 + R_0/Z_2 + Z_1/Z_2}\right) e, \quad e'' = \left(\frac{2}{1 + R_0/Z_2 + Z_1/Z_2}\right) e$$

several microseconds—approximately 6 ms. in the last wave on the left in the figure.

Fig. 19. Junction of Two Lines.—When the incident wave reaches the junction, a part is reflected back and a part is transmitted on to the other line. The relative division depends upon the ratio of the surge impedances of the two lines. If $Z_2 < Z_1$, the reflection is negative and $e'' < e$. If $Z_2 = Z_1$, there is no reflection and the full wave is transmitted, $e'' = e$. If $Z_2 > Z_1$, the reflection is positive and $e'' > e$, but can not exceed $2\,e$. As far as conditions at the junction are concerned, the surge impedance Z_2 could just as well be replaced by a resistance $R_2 = Z_2$, and the equations are identical with those of Fig. 13. This case has been illustrated with particular waves, but incident waves of any shape are reflected from, and transmitted across, a junction between two lines, without change of shape. The waves in this figure have been drawn on a *time axis*, and so the reflected and transmitted waves are of equal length.

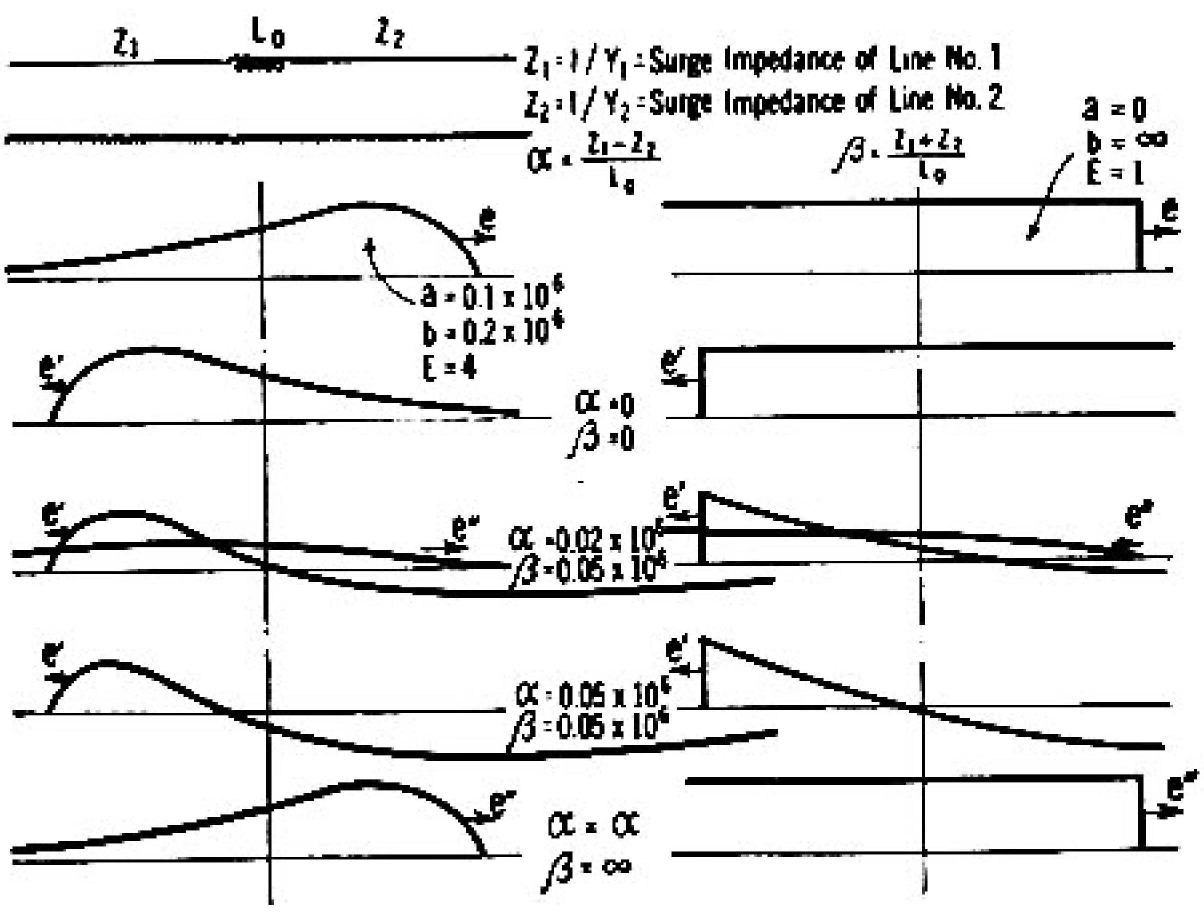


FIG. 22.—Two Lines Connected by an Inductance L_0

$$e = E\,(\epsilon^{-at} - \epsilon^{-bt})$$

$$e' = E\left[\frac{a+\alpha}{a-\beta}\epsilon^{-at} - \frac{b+\alpha}{b-\beta}\epsilon^{-bt} + \frac{(\alpha+\beta)\,(a-b)}{(a-\beta)\,(b-\beta)}\epsilon^{-\beta t}\right]$$

$$e'' = E\left[\frac{\alpha-\beta}{a-\beta}\epsilon^{-at} - \frac{\alpha-\beta}{b-\beta}\epsilon^{-bt} + \frac{(\alpha-\beta)\,(a-b)}{(a-\beta)\,(b-\beta)}\epsilon^{-\beta t}\right]$$

Fig. 20. Junction of *N* Lines.—Incident waves of any shape are transmitted and reflected without change of shape. Incident, reflected, and transmitted waves have the same length on a *time* axis,

but are contracted on a *space* axis proportional to their respective velocities of propagation; for

$$t = \frac{x_1}{v_1} = \frac{x_2}{v_2} = \ldots = \frac{x_n}{v_n}$$

therefore $\quad x_1 : x_2 : x_3 \ldots = v_1 : v_2 : v_3 \ldots$

and of course, $\quad v = 1/\sqrt{LC}$

This contraction is illustrated for finite rectangular waves, but applies to waves of all shapes.

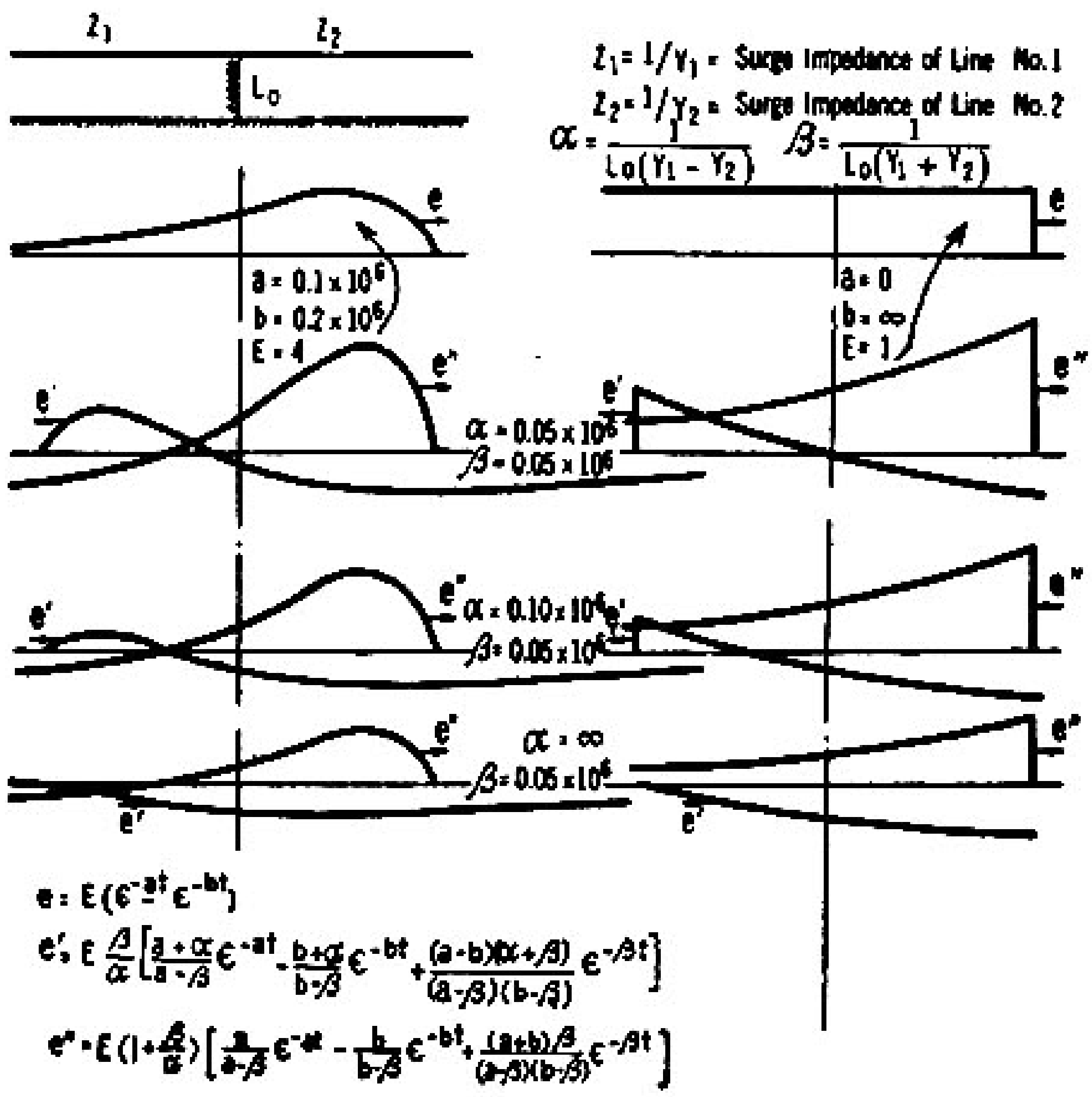


FIG. 23.—Two Lines with Shunt Inductance at Their Junction

Fig. 21. Two Lines Connected by a Resistor R_0.—As in the two previous cases there is no distortion of wave shape, but the resistor consumes part of the energy. By making $R_0 = (Z_1 - Z_2)$, it is possible to wipe out the reflected waves. The curves show how the waves and their energy content vary with R_0, Z_1, and Z_2.

Fig. 22. Two Lines Connected by an Inductance L_0.—The reflected wave exhibits the same characteristics as in the case of Fig. 14. If L_0 is large enough, the major portion of the incident wave can be reflected back, and only a small transmitted wave passed

through. However, the conventional choke coil of a few years ago was entirely inadequate in this respect, being too small to affect the transmitted wave by more than a few per cent. For example, taking $Z_1 = Z_2 = 500$ ohms, and $L_0 = 33$ microhenrys (a common standard for choke coils), and an incident wave

$$e = E\left(\varepsilon^{-at} - \varepsilon^{-bt}\right) = E\left(\varepsilon^{-0.1t} - \varepsilon^{-0.2t}\right)$$

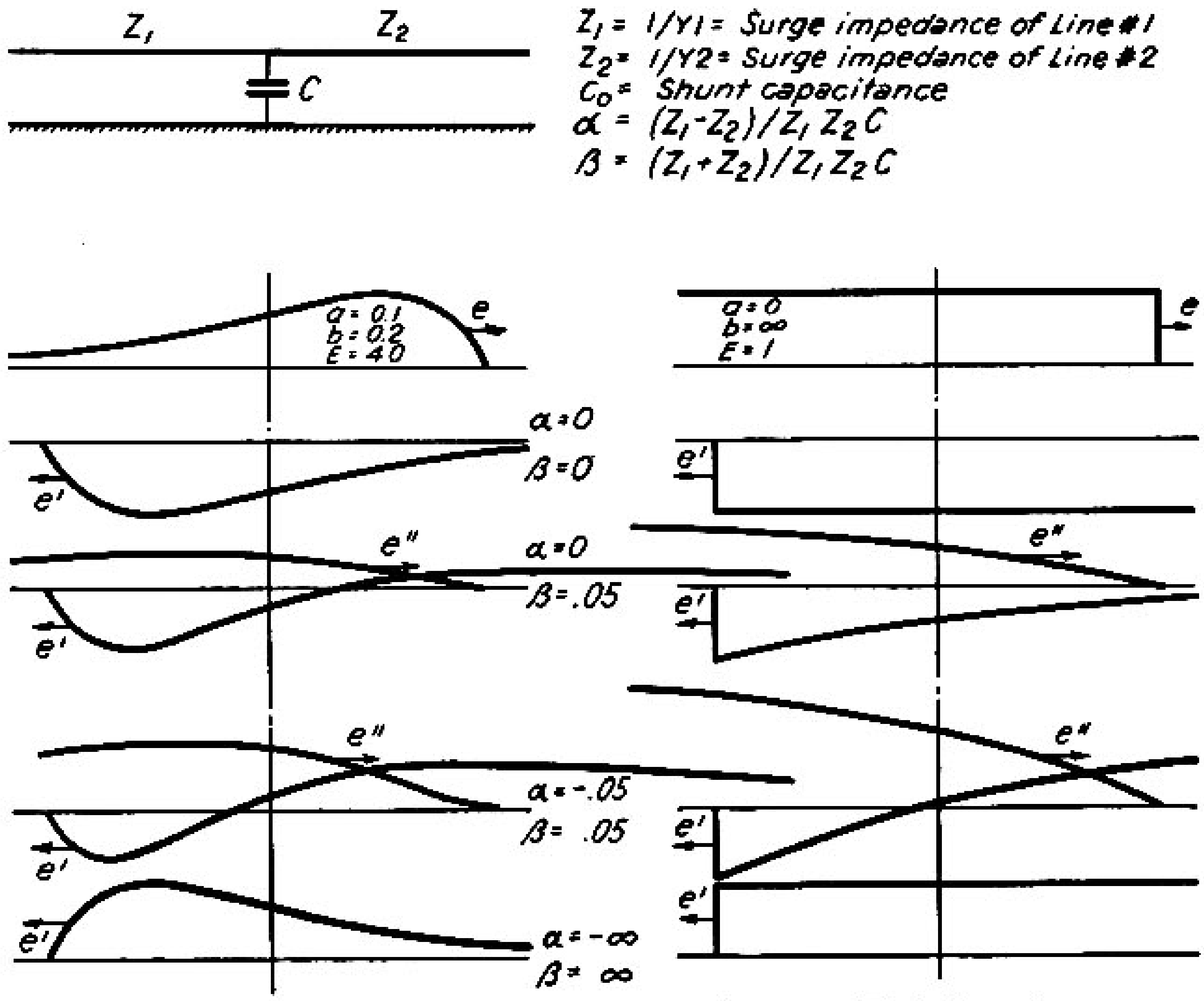


FIG. 24.—Two Lines with Shunt Capacitance at Their Junction

$$e = E\left(\epsilon^{-at} - \epsilon^{-bt}\right)$$

$$e' = E\left[-\frac{a-\alpha}{a-\beta}\epsilon^{-at} + \frac{b-\alpha}{b-\beta}\epsilon^{-bt} + \frac{(a-b)(\alpha-\beta)}{(a-\beta)(b-\beta)}\epsilon^{-\beta t}\right]$$

$$e'' = E\left[\frac{\alpha-\beta}{a-\beta}\epsilon^{-at} - \frac{\alpha-\beta}{b-\beta}\epsilon^{-bt} + \frac{(a-b)(\alpha-\beta)}{(a-\beta)(b-\beta)}\epsilon^{-\beta t}\right]$$

which has a 7-ms. front and a 20-ms. tail, there results

$$e'' = E\left[\frac{30.0}{29.9}\varepsilon^{-0.1t} - \frac{30.0}{29.8}\varepsilon^{-0.2t} + \frac{0.1 \times 30}{29.8 \times 29.9}\varepsilon^{-30t}\right]$$

$$\cong E\left(\varepsilon^{-0.1t} - \varepsilon^{-0.2t}\right)$$

Thus such a small series inductance is entirely ineffective against natural lightning waves, and the entire incident wave is transmitted with negligible change of shape. Of course, the effect is more pronounced for waves with steeper fronts, but it is never of practical importance. The choke coil is discussed in greater detail in Chapter V.

Fig. 23. Two Lines with Shunt Inductance at Their Junction.—The equations and curves for the incident and reflected waves in this case hold identically, when Z_2 is replaced by R_0 to ground, and the circuit reverts to that of Fig. 16.

Fig. 24. Two Lines with Shunt Capacitance at Their Junction.—The effect of a capacitance in shunt is similar to that of an inductance in series. This use of capacitance is often proposed in connection with protective schemes for power systems, but it is economically limited to low-voltage circuits, because the requisite amount of capacitance goes up as the square of the voltage. Thus, doubling the voltage necessitates twice as many capacitor units in series in order to support the stress, and thereby requires twice as many stacks in parallel to give the same capacitance. However, capacitors in shunt are quite feasible for the protection of generators; they are discussed in more detail in Chapter V.

SUMMARY OF CHAPTER III

The traveling-wave analysis developed in Chapter I is applied to a large number of specific cases, which cover many practical examples on transmission systems. Equations for the reflected and transmitted waves are given and the corresponding graphs are plotted so that the principal characteristics of the different transition points are evident. Most transition points of practical importance may be classified under one of three groups:

a. Those which do not involve a change of wave shape.

b. Those which give rise to exponential terms.

c. Those which give rise to damped oscillatory terms.

However, any transition point for which the differential equations have a known solution can be calculated. The procedure is simple and straightforward, and the agreement with cathode-ray oscillograms is surprisingly good.

CHAPTER III

ATTENUATION AND DISTORTION

As a traveling wave moves along the line it suffers three different changes: (*a*) the crest of the wave decreases in amplitude, or is *attenuated*; (*b*) the wave changes shape, that is, becomes more elongated, its irregularities are smoothed out, and its steepness is reduced; (*c*) the potential and current waves cease to be similar. The latter two changes occur together and are called *distortion*. It is theoretically possible to have attenuation without distortion, as in Heaviside's distortionless line, where $RC = GL$. But the converse is not true, for distortion is always accompanied by attenuation. As pointed out in Chapter I, this distortionless feature is practically realized in the loaded telephone circuit. Attenuation and distortion are caused by energy losses, and these are due to the conductor resistance as modified by transient skin effect, to leakage over the insulators, to dielectric losses, and to corona. The latter is by far the most important, as far as high-voltage surges are concerned. Artificial lightning surges put on transmission lines sustain their shape, and attenuate very slowly if below the corona voltage, but attenuate rapidly and become badly distorted if high above the corona voltage.

Very little is known about the mechanism of corona loss, although suitable empirical formulas have been devised for computing the loss under power frequency conditions. Whether or not the corona loss continues to vary as the square of the difference between the surge voltage and the critical corona voltage, under transient conditions, has not yet been established; nor is it known that the critical corona voltage is the same under the two conditions. Thus, consideration of the attenuation and distortion of traveling waves hinges upon a subject about which little is known; as a result, the calculation of attenuation is only on a rough empirical basis. Moreover, there does not appear to be any hope of deriving rational attenuation formulas until the nature of corona under surge conditions is established.

An interesting speculation on the effect of corona in distorting the wave has been given by E. W. Boehne.* Referring to Fig. 25, the

* Discussion, *A.I.E.E. Trans.*, Vol. 50, p. 558.

traveling wave is divided into a number of laminations corresponding to different voltage levels, and each voltage level is assumed to extend the conducting corona region by a proportionate amount. Now if these conducting corona regions increase the capacitance to ground, but do not change the inductance, that is are conducting radially but not axially, then to each voltage level there corresponds a different velocity of propagation:

$$v_k = \frac{1}{\sqrt{LC_k}}$$

and therefore the top laminations, traveling at slower speeds, will slip back, decapitating the crest, slowing the front, and filling in the tail of the wave as shown in Fig. 25. This explanation agrees with observations of both natural and artificial lightning surges, and offers a

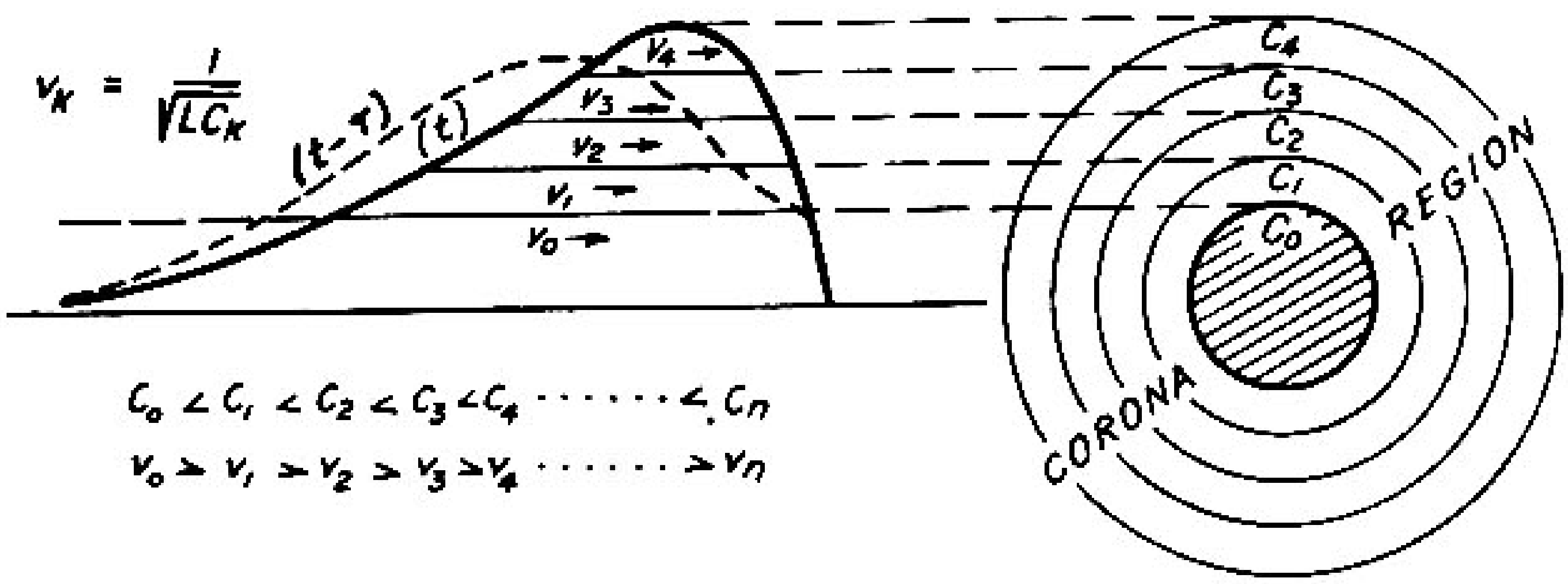


FIG. 25.—Effect of Corona on Wave Shape

plausible reason for the shearing back of the wave front above a particular level which seems to be the critical corona voltage.

Another explanation * of the part played by corona in causing distortion has to do with the charge which enters the corona envelope and reduces the voltage on the front of the wave as it rises above the critical voltage. When the voltage begins to fall the charge returns to the conductor, and thereby builds up the voltage on the tail of the wave. This exchange of charge between the conductor and the corona envelope is accompanied by a loss of energy, but not of charge.

Attenuation due to corona is greater for positive polarity waves than for negative. This is because positive corona loss is greater for a given voltage than negative. For equal potential waves on several conductors the attenuation is less than on a single conductor, because

* "Experimental Studies in the Propagation of Lightning Surges on Transmission Lines," O. Brune and J. R. Eaton, *A.I.E.E. Trans.*, Vol. 50, p. 1132.

the gradient at the surface of the conductor is less and therefore the corona is not so intense. Ground wires increase the attenuation at high voltage, and decrease it at low voltage; for at high voltage, corona appears on both the line and ground wires, whereas at low voltages it appears only on the line wire and the ground wire reduces the ground resistance. However, neither effect is very pronounced.

On the assumption that the voltage and current waves remain similar, and in the ratio of the surge impedance of the line, it is possible to formulate a simple differential equation relating the voltage to the rate of energy loss, and therefrom to derive expressions for the voltage corresponding to different laws of energy loss.

Consider a traveling wave $e = f(x - vt)$ of length D, see Fig. 26,

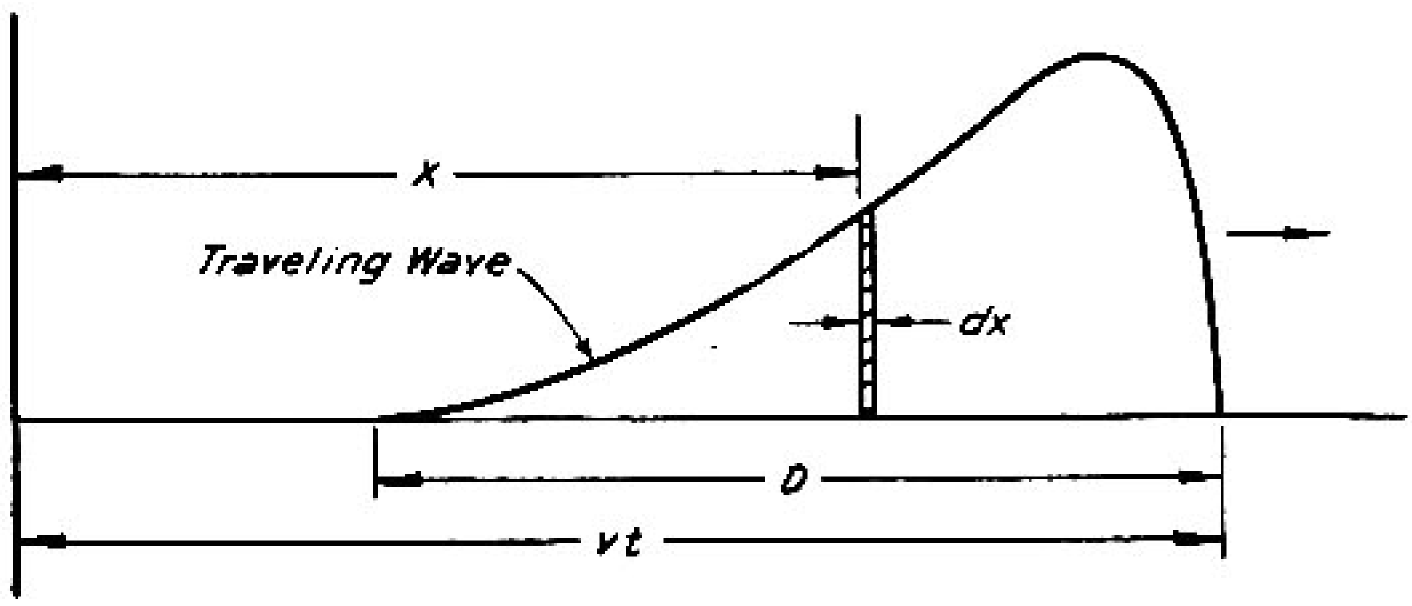


FIG. 26

and its companion current wave i, and suppose that these two waves are related by the surge impedance:

$$e = Zi = \sqrt{\frac{L}{C}}\, i \tag{69}$$

At time t the toe of the wave is at $x = vt$ from the origin, and the total stored energy of the wave is

$$W = \frac{C}{2}\int_{(vt-D)}^{vt} e^2 dx + \frac{L}{2}\int_{(vt-D)}^{vt} i^2 dx = C\int_{(vt-D)}^{vt} f^2\,(x - vt)\,dx \tag{70}$$

The rate at which the energy content is changing with respect to t is

$$\frac{dW}{dt} = C\int_{(vt-D)}^{vt} \frac{\partial}{\partial t} f^2\,(x - vt)\cdot dx + C\,v\,f^2\,(vt - vt) - Cv\,f^2(vt - D - vt)$$

$$= C\int_{(vt-D)}^{vt} \frac{\partial}{\partial t} f^2\,(x - vt)\cdot dx = C\int_{(vt-D)}^{vt} \frac{\partial e^2}{\partial t}\,dx \tag{71}$$

because $f\,(0) = 0$ and $f\,(-D) = 0$

Now if the rate of energy dissipation due to the line losses is a function of the voltage $\phi\ (e)$, then the total rate of energy loss for the entire wave is

$$\int_{(vt-D)}^{vt} \phi\ (e) \cdot dx \tag{72}$$

Equating (71) and (72) and discarding the integrals

$$C \frac{\partial\ (e^2)}{\partial t} = -\ \phi\ (e) \tag{73}$$

subject to the initial conditions

$$e = E\ (x) \quad \text{at} \quad t = 0 \tag{74}$$

The solution of (73) thus defines the wave at any instant t. By way of illustration, four special cases will be considered.

Case I. Ideal Transmission Line.—The ideal transmission line is characterized by four line constants R, L, C, G, and if Equation (69) holds, the rate of energy loss is

$$\phi(e) = Ri^2 + G\,e^2 = R \frac{C}{L} e^2 + G\,e^2 = \left(\frac{RC + LG}{L}\right) e^2 \tag{75}$$

Substituting (75) in (73) and using (74) there results

$$e = E\varepsilon^{-\alpha t} \tag{76}$$

where

$$\alpha = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right) \tag{77}$$

The attenuation, or rate of decay, is

$$\frac{de}{dt} = -\ \alpha E \varepsilon^{-\alpha t} = -\ \alpha e \tag{78}$$

Case II. The Skilling Formula.*—If the loss is assumed to vary as the excess voltage above the critical corona voltage e_0, then

$$\phi\ (e) = \beta\ (e - e_0) \tag{79}$$

and (73) and (74) give

$$\frac{\beta}{2\ C} t = at = (E - e) + e_0 \log \left(\frac{E - e_0}{e - e_0}\right) \tag{80}$$

$$\frac{de}{dt} = -\ a\left(1 - \frac{e_0}{e}\right) \tag{81}$$

* "Corona and Line Surges," by H. H. Skilling, *Electrical Engineering*, October, 1931. See also "Letter to the Editor" in *Electrical Engineering* for November, 1931.

Case III. The Quadratic Formula.—If the loss is assumed to vary as the square of the excess voltage above the critical corona voltage, then

$$\phi(e) = \gamma (e - e_0)^2 \tag{82}$$

Then by (73) and (74)

$$\frac{t\gamma}{2C} = bt = \frac{(E - e)\, e_0}{(E - e_0)(e - e_0)} + \log\left(\frac{E - e_0}{e - e_0}\right) \tag{83}$$

$$\frac{de}{dt} = \frac{-b\,(E - e_0)(e - e_0)^2}{e_0\,(E - e)} \tag{84}$$

Case IV.—The Foust and Menger Formula.*—If the loss is assumed to vary as the cube of the voltage, then

$$\phi(e) = \lambda e^3 \tag{85}$$

and by (73) and (74)

$$e = \frac{E}{\dfrac{\lambda}{2C} Et + 1} = \frac{E}{KEt + 1} \tag{86}$$

This expression was originally given as an empirical formula to fit observed attenuation data; and its parameter K was found to range from 0.02×10^{-6} to 0.14×10^{-6} for different lines and conditions, E being in *volts* and t in *microseconds*. Differentiating

$$\frac{de}{dt} = \frac{-KE^2}{(KEt + 1)^2} = -\,Ke^2 \tag{87}$$

thus showing that the attenuation, or rate of decay, is proportional to the square of the voltage in this case.

It is interesting and instructive to make a comparison between the above four formulas for the surge voltage as function of the time or distance of travel. In Fig. 27 these four different formulas have been plotted for a 2000-kv. surge, so as to pass through a common point at 50 per cent of the initial voltage of the surge. It is convenient to consider three principal regions.

Region I—from the initial voltage of $E = 2000$ kv. to the 50 per cent voltage point of 1000 kv.

* "Surge Voltage Investigation of Transmission Lines," by W. W. Lewis, *A.I.E.E. Trans.*, 1928.

Region II—from $e = 1000$ kv. to the critical corona voltage $e_0 = 500$ kv.

Region III—below the critical corona voltage $e_0 = 500$ kv.

In Region I, the Foust and Menger formula agrees almost perfectly with the quadratic formula, and Skilling's formula agrees equally well with the exponential law. The difference between these two pairs is not great, and of little practical importance. In this region of greatest interest all four formulas are of practically equal accuracy,

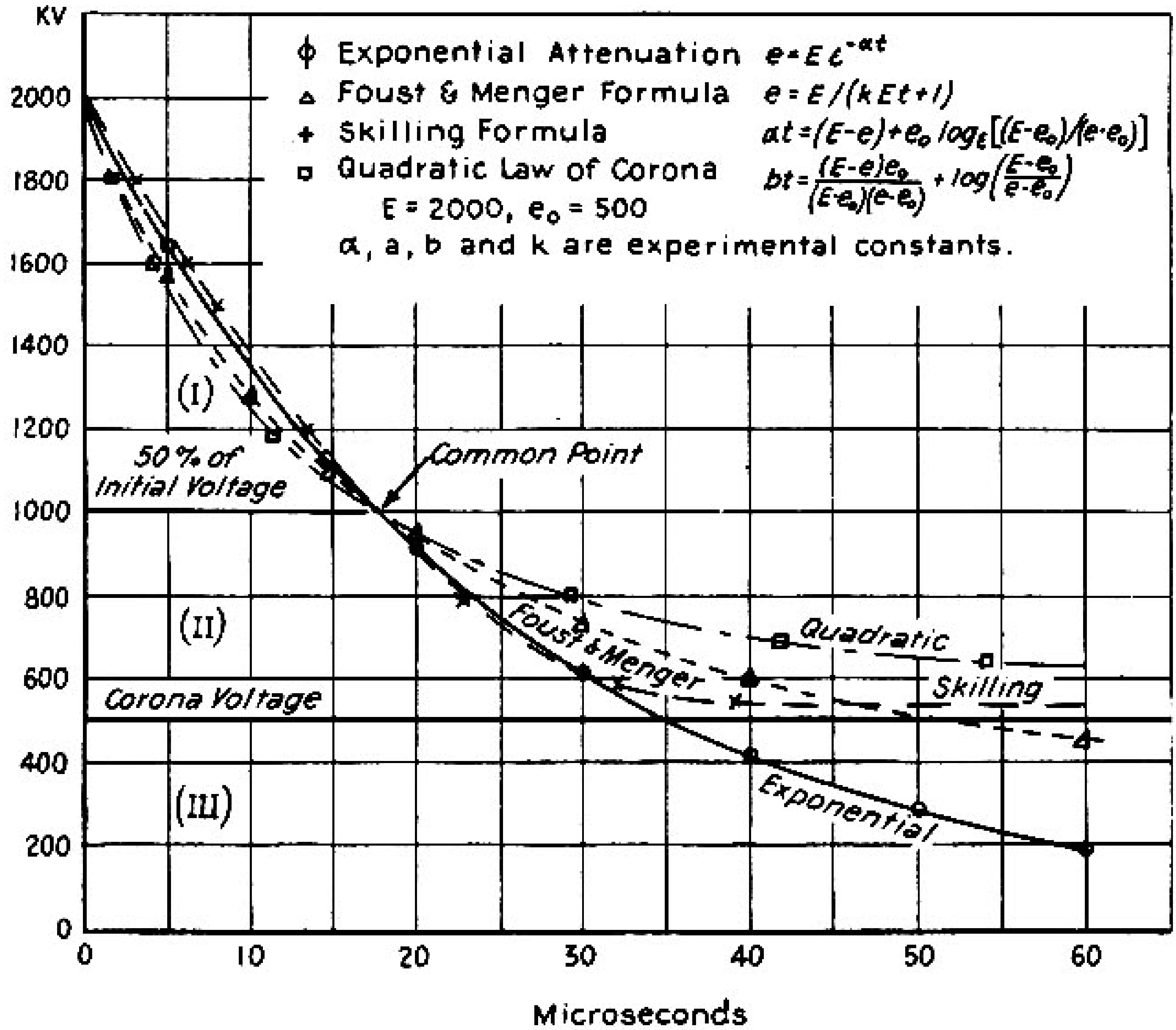


FIG. 27.—Attenuation Formulas

and therefore the formula will be used which is the most convenient. The Foust and Menger formula is the most simple for estimating the attenuation, but the exponential formula is easier to operate upon mathematically.

In Region II the Foust and Menger formula parts company with the quadratic formula, but the Skilling formula and exponential law continue to agree until the critical corona voltage is neared, when the Skilling formula abruptly flattens out while the exponential crosses to Region III.

Both the Skilling and the quadratic formulas are asymptotic to the critical corona voltage, and therefore do not appear in Region III. The Foust and Menger and the exponential formulas enter Region II, but at widely different points, and diverge considerably.

Although these formulas appear on the surface to have been derived by a rational process, it should be noticed that all of them are based on the assumption that the current and voltage waves are exact replicas of each other—in other words, the distortion is ignored. Moreover, the experimental constants α, a, b, and K have to be determined from tests on the transmission line in question, and under the actual conditions that are to prevail. Thus none of these formulas can be used to predict attenuation until the constants have been ascertained for the particular conditions and line in question. The very abrupt flattening of the Skilling formula as the critical corona voltage is approached is not always evidenced by experimental data; and of course, the failure of both the Skilling and the quadratic formulas to cross into Region III is contrary to actual facts.

Influence of Ground Wires on Attenuation.—The presence of a ground wire necessitates a higher charge on a line conductor to maintain the same potential. Therefore the gradient at the surface of the conductor is higher and (if above the critical value) increases the corona, so that traveling waves of a given initial voltage are attenuated more rapidly on lines equipped with ground wires than on those without. On the other hand, if corona does not form, the ground wires may actually decrease the attenuation by lowering the effect of the ground resistance. However, neither of the above effects is very decisive.

SUMMARY OF CHAPTER III

The effect of corona and transient skin effect in distorting and attenuating traveling waves is discussed briefly, and it is pointed out that corona is the principal cause in this respect, as far as high-voltage surges are concerned. On the assumption that corona increases the capacitance of the conductor without offering a corresponding increased diameter to the flow of current in the direction of the conductor, it is easy to account for the peculiar distortion experienced by high-potential traveling waves, for parts of the wave above the critical corona voltage must travel at a slower rate in accordance with the relationship

$$v = \frac{1}{\sqrt{L\,C}}$$

Therefore the top laminations of the wave, traveling at slower speeds, will slip back, decapitating the crest, slowing the front, and filling in the tail of the wave.

If it is assumed that the potential and current waves remain similar during attenuation, the differential equation defining attenuation is

$$\frac{\partial (e^2)}{\partial t} = - \phi (e)$$

where $\phi (e)$ is the function expressing the rate of energy loss. Corresponding to different assumptions as to the nature of $\phi (e)$ there are derived four formulas for attenuation: ideal line, Skilling's formula, the quadratic formula, and the Foust and Menger formula. Down to half voltage there is little to choose between any of these formulas, but all of them depend upon empirical constants.

There is perhaps a greater need for a reliable and comparatively simple attenuation formula than for any other single item in traveling-wave theory.

CHAPTER IV

SUCCESSIVE REFLECTIONS

In many important problems, such as in the theory of ground wires, the effect of short lengths of cable, trunk lines tapped at intervals, and the process of charging or discharging a line, it is necessary to consider the successive reflections of traveling waves. Sometimes it is exceedingly difficult to keep track of the multiplicity of these successive reflections, so a lattice, or time-space diagram,* has been devised which shows at a glance the position and direction of motion of every incident, reflected, and refracted wave on the system at every instant of time. In addition, this lattice provides the means for calculating the shape for all reflected and refracted waves and gives a complete history of their past experience. Even the effects of attenuation and wave distortion can be entered on the lattice, if the defining functions are known.

The principle of the reflection lattice is illustrated in Fig. 28. Three junctions, Nos. 1, 2, and 3, placed at unequal intervals along the line, are shown. These junctions may consist of any combinations of impedances in series with the line or shunted to ground. In fact, no restrictions are placed on the generality of the impedances at the junctions as far as the lattice is concerned, although their complexity may preclude a mathematical solution of the differential equations which the lattice gives. The circuits between junctions may be either overhead lines or cables, having, in general, different surge impedances, velocities of wave propagation, and attenuation factors. To construct the lattice, lay off the junctions to scale at intervals equal to the times of passage of the wave on each section between junctions. Then choose a suitable vertical time scale, shown in Fig. 28 at the left of the lattice, and draw in the diagonals. The great advantage of laying off the junctions at intervals equal to the time of wave passage instead of to the actual lengths between junctions is that the diagonals all have the same slope, and the time scale is applicable to every branch. At the top of the lattice, at any convenient place centered on the junctions, place indicators with the reflection and refraction operators marked on them. In the notation

* Discussion by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 49, 1930.

of Fig. 28 these indicators are shown as little double-headed arrows labeled as follows:

a = reflection operator for waves approaching from the left.
a' = reflection operator for waves approaching from the right.
b = refraction operator for waves approaching from the left.
b' = refraction operators for waves approaching from the right.
α = attenuation factor for section between junctions.

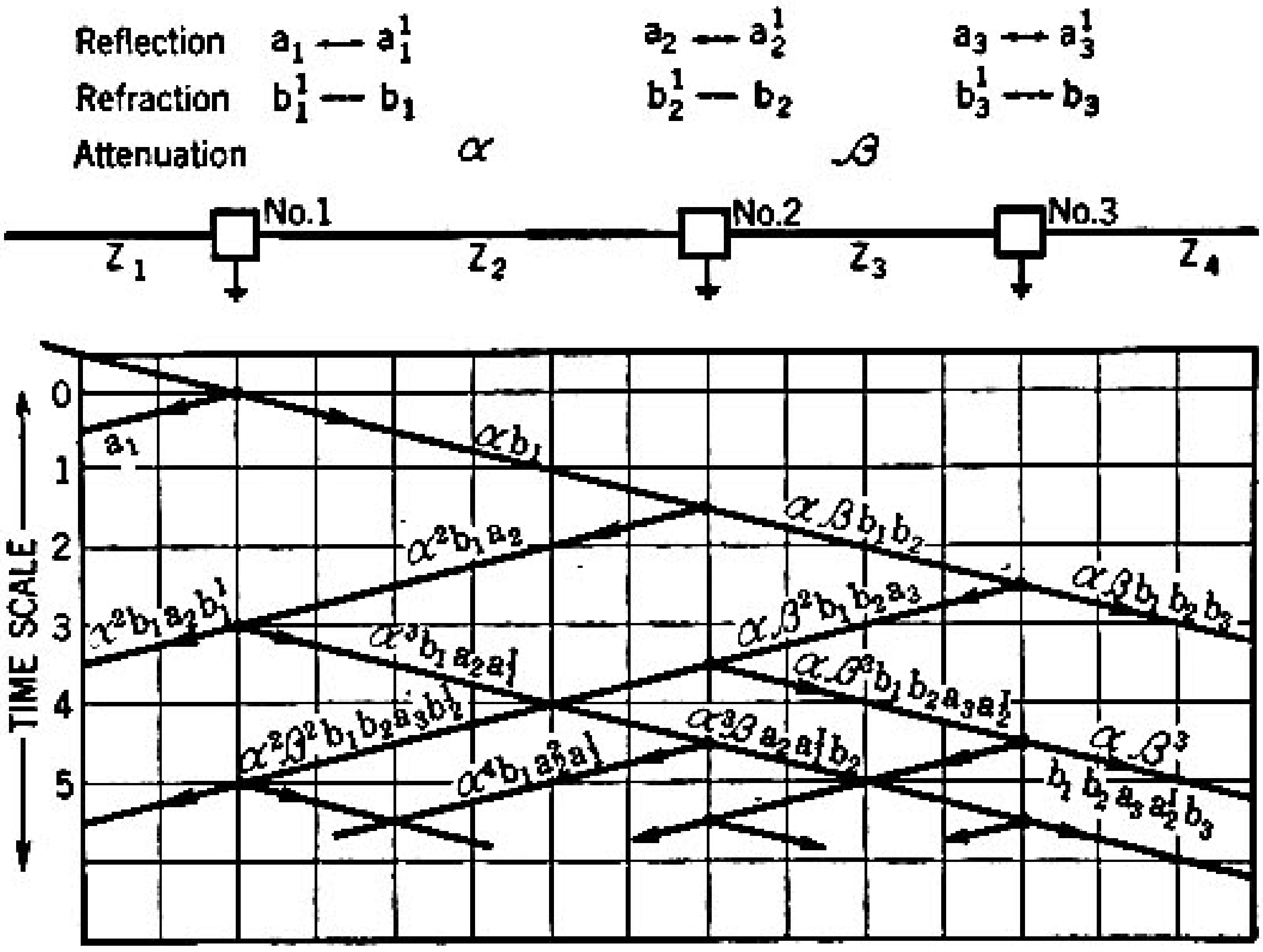


FIG. 28.—Lattice for Calculating Successive Reflections

Now starting at the origin of the initial incident wave at the upper left-hand corner of the lattice, obtain the operators for the reflected and refracted waves at each junction by applying the reflection and refraction operators at that junction to the incident waves arriving there from both the left and the right, and proceed until the lattice is completed. It will be observed that:

1. All waves travel downhill.
2. The position of any wave at any time is given by the time scale at the left of the lattice.
3. The total potential at any point at any instant of time is the superposition of all the waves which have arrived at that point up until that instant of time, displaced in position from each other by intervals equal to the differences in their time of arrival.
4. The previous history of any wave is easily traced, that is, where it came from, and just what other waves went into its composition.

5. Attenuation is included, so that the amount by which a wave is reduced in traveling between junctions is taken into account.
6. If it is desirable to carry the computations to a point where it is not practical to place the various operators directly on the lattice itself, then the arms may be numbered, and the corresponding operational expressions tabulated in a suitable table. It is sometimes possible to devise purely tabular methods which can be filled in automatically. If the junctions contain only resistances, then it is most simple to fill in the numerical values directly on the lattice.

The use of the lattice will be more fully appreciated after a few typical examples of its application are studied in detail.

Charging of a Line from a D-C. Source.—Fig. 29 illustrates the part played by the attenuation and successive reflections in the charging of an open-ended transmission line from a fixed d-c. source of infinite capacity. The reflection operator at the open end of the line is $(+1)$ and at the generator end it is (-1) because the voltage is maintained there at a constant value and so any wave returning to the source is immediately nullified by an equal and opposite wave. For the sake of simplicity the attenuation has been made linear and equal to 50 per cent. Without attenuation, the cycle of oscillations repeats indefinitely; but when line losses are present the oscillations gradually diminish until the line eventually reaches a steady-state condition. However, a line possessing both leakage and series resistance can never become fully charged to the terminal potential, throughout its length, for the flow of current due to the conductance to ground causes a progressive voltage drop in the resistance of the line. Therefore, the ultimate level charging of a line requires that there be no leakage currents. Referring to Fig. 29, the voltage at various instants of time for an attenuation $(1 - \alpha)$ per trip is:

-1 Sending End	Units of Time	$+1$ Receiving End
1	0	0
1	1	α
$1 + \alpha^2$	2	2α
1	3	$2\alpha - \alpha^3$
$1 - \alpha^4$	4	$2\alpha - 2\alpha^3$
1	5	$2\alpha - 2\alpha^3 + \alpha^5$
$1 + \alpha^6$	6	$2\alpha - 2\alpha^3 + 2\alpha^5$
1	7	$2\alpha - 2\alpha^3 + 2\alpha^5 - \alpha^7$
$1 - \alpha^8$	8	$2\alpha - 2\alpha^3 + 2\alpha^5 - 2\alpha^7$
etc.		etc.

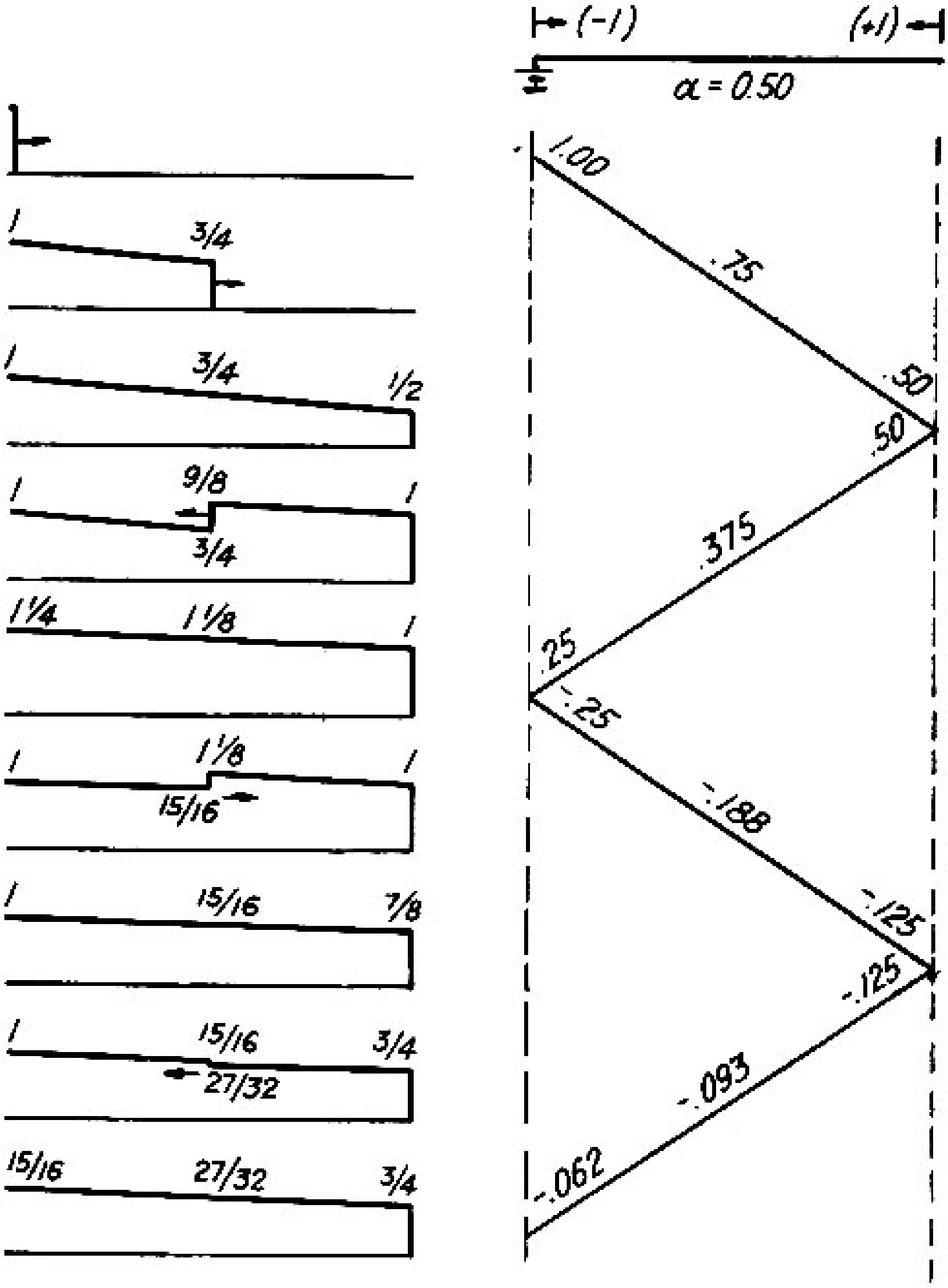


FIG. 29.—Charging a Line from an Infinite D–C Source

The voltage at the receiving end after 4 n units of time is

$$
\begin{aligned}
e &= 2\,(\alpha - \alpha^3 + \alpha^5 - \alpha^7 + \ldots) \\
&= 2\,(\alpha - \alpha^3)\,(1 + \alpha^4 + \alpha^8 + \ldots) \\
&= 2\,\alpha\,(1 - \alpha^2) \sum_0^n \alpha^{4r} \\
&= 2\,\alpha\,\frac{1 - \alpha^{4(n+1)}}{1 + \alpha^2} \qquad (88)
\end{aligned}
$$

After an infinite number of oscillations

$$
e = \frac{2\,\alpha}{1 + \alpha^2} \qquad (89)
$$

Thus if the attenuation is $(1 - \alpha) = 0.5$, and the line is distortionless, the open end of the line finally stabilizes at a voltage of

$$e = \frac{2 \times 0.5}{1 + 0.25} = 0.8$$

Had the line been grounded through a resistance R its potential would have eventually stabilized, even though the line were absolutely free of losses. This can be easily seen by making $\alpha = 1$ and placing on the lattice of Fig. 29 the reflection operator

$$a = \frac{R - Z}{R + Z} < 1 \qquad (90)$$

Then
$$\begin{aligned} e &= (1 + a)(1 - a + a^2 - a^3 + \ldots) \\ &= (1 - a^2)(1 - a^2 + a^4 + \ldots) \\ &= (1 - a^2)\frac{1 - a^{2(n+1)}}{1 - a^2} = 1 - a^{2(n+1)} = 1 \text{ as } n \to \infty \end{aligned} \qquad (91)$$

Charging a Line from an Impulse Generator.*—The impulse generator used for studying the effect of artificial lightning surges on transmission lines, or on insulators and power apparatus, consists of a capacitor charged to some voltage E and arranged to discharge through an impedance such that a voltage wave of arbitrary shape is impressed upon the circuit under test. The simplest form of the impulse generator is a capacitor arranged to discharge directly into the circuit under test. Fig. 30 illustrates an impulse generator connected to a transmission line with both an open and grounded end. The reflection operators are:

$+1$ = reflection operator for an open end.

-1 = reflection operator for a grounded end.

$a = -\dfrac{p - \beta}{p + \beta}$ = reflection operator at capacitor, Fig. 10E, in which $\beta = 1\ Cz$.

$b = 1 + a = \dfrac{2\beta}{p + \beta}$ = refraction operator at capacitor.

* "Attenuation and Successive Reflections of Traveling Waves," by J. C. Dowell, *A.I.E.E. Trans.*, Vol. 50, p. 551.

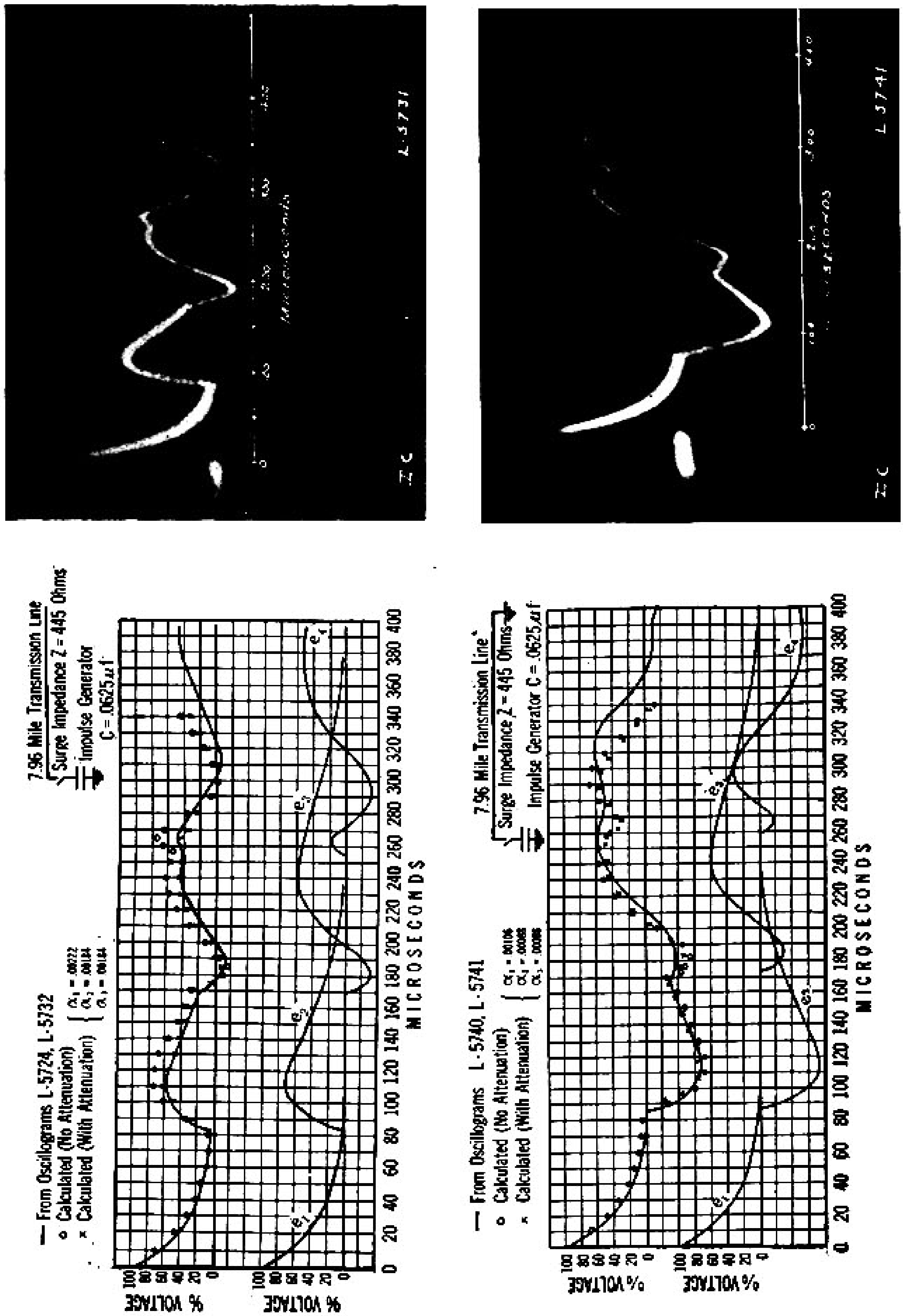


FIG. 30.—Charging a Line from an Impulse Generator

It is this refraction operator which gives the total voltage at the capacitor for each incident wave.

The first wave due to the initial discharge of the capacitor into the surge impedance Z of the transmissión line is exactly the same as the discharge of a charged capacitor into a resistor $R = Z$, the well-known equation for which is

$$e_1 = E\varepsilon^{-\beta t}$$

The subsequent voltages at the impulse generator due to successive reflections then are, for the open-end line

$$\left.\begin{aligned} e_2 &= \alpha^2 b e_1 &&= 2\,\alpha^2 E\beta t\varepsilon^{-\beta t} \\ e_3 &= \alpha^4 a b e_1 &&= 2\,\alpha^4 E\beta t\,(\beta t - 1)\,\varepsilon^{-\beta t} \\ e_4 &= \alpha^6 a^2 b e_1 &&= \frac{2}{3}\,\alpha^6 E\beta t\,(2\,\beta^2 t^2 - 6\,\beta t + 3)\,\varepsilon^{-\beta t} \\ &\ldots \end{aligned}\right\} \tag{92}$$

The free traveling waves leaving the capacitor are:

$$\left.\begin{aligned} e_1 &= E\varepsilon^{-\beta t} \\ \alpha^2 a e_1 &= \alpha^2 E\,(2\,\beta t - 1)\,\varepsilon^{-\beta t} \\ \alpha^4 a^2 e_1 &= \alpha^4 E\,(2\,\beta^2 t^2 - 4\,\beta t + 1)\,\varepsilon^{-\beta t} \\ \alpha^6 a^3 e_1 &= \alpha^6 E\left(\frac{4}{3}\,\beta^3 t^3 - 6\,\beta^2 t^2 + 6\,\beta t - 1\right)\varepsilon^{-\beta t} \\ &\ldots \end{aligned}\right\} \tag{93}$$

A similar set of equations for the voltages at the impulse generator, and the corresponding free traveling waves, obtain for the grounded-end line. The voltage components have been plotted in Fig. 30 when the attenuation is negligible, for both open-end and grounded-end conditions, and the total voltage at the impulse generator is found by superimposing the overlapping parts of each component wave. By a direct comparison between the calculated voltages and the observed waves recorded by the cathode-ray oscillograph, the attenuation factor α can be estimated. But the attenuation is dependent upon the voltage and therefore will be different for each reflection, so that it would have been more rational to use values of $\alpha_1, \alpha_2, \alpha_3, \ldots$ instead of the same α for each reflection. This method is not to be recommended for determining the attenuation, because the wave shape changes at each reflection, and the calculations ignore the distortion.

Fig. 30*A* shows an impulse generator at an intermediate point on the line, the corresponding lattice, and an oscillogram of an actual case. This example is interesting chiefly because the lattice shows that at the fifth transition the voltage at the impulse generator will double up, for waves arrive simultaneously from the right and left; and this prediction is verified by the cathode-ray oscillogram. The equations for this case are given in Dowell's paper, but his notation is different from that used in this book.

It will be observed that these successive reflections exhibit the following characteristics:

a. Reduced maximum crests.
b. Increased lengths.
c. Oscillatory components.

The lengthening of the waves at each reflection is responsible for the piling up of waves at the impulse generator, so that after a few reflections there are always several waves at the transition point simultaneously.

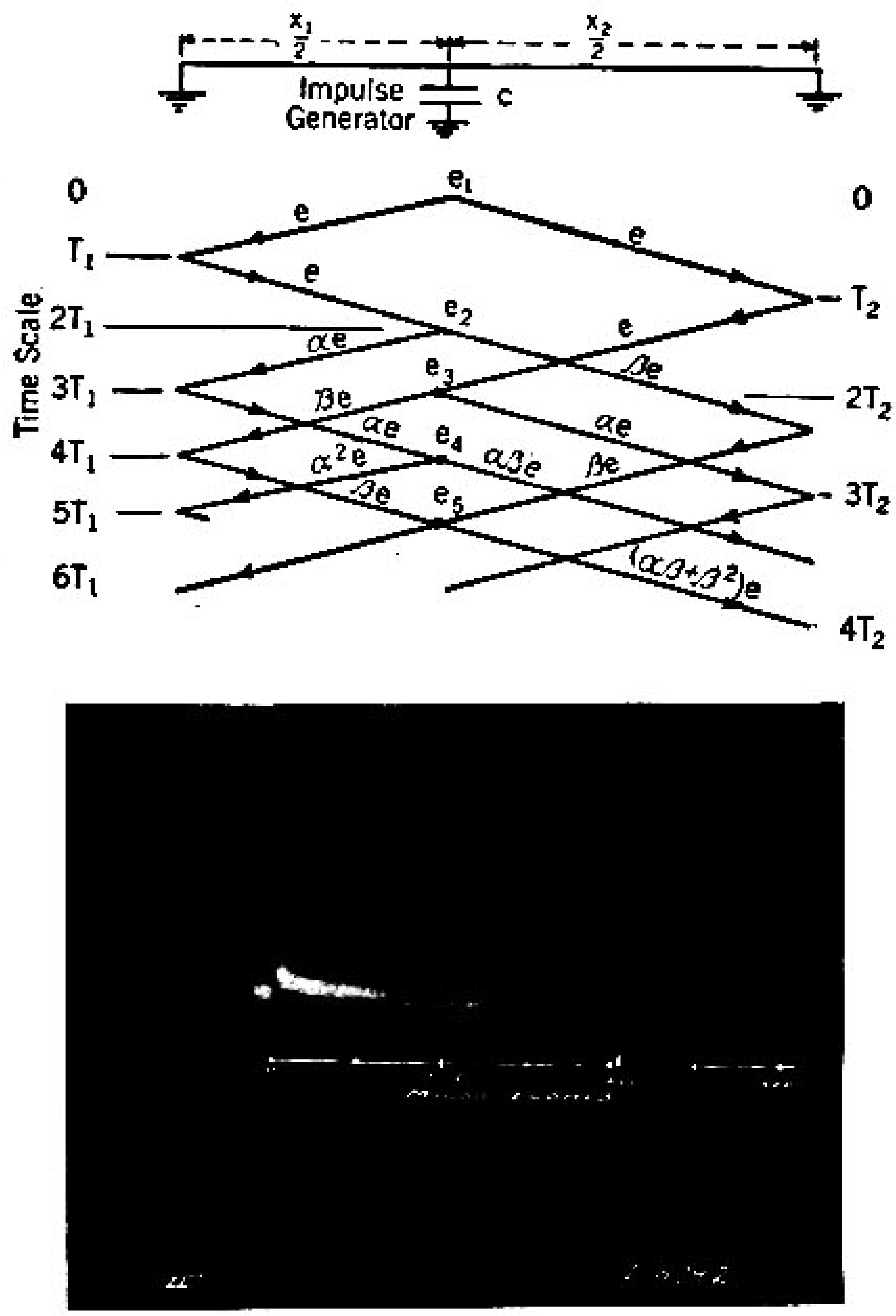


FIG. 30*a*.—Successive Reflections from Impulse Generator at Intermediate Point on Line

Reflections between a Capacitor and a Resistor.—The mechanism of successive reflections in eliminating the effect of the surge impedance of a transmission line is beautifully illustrated by this case. The lattice is shown in Fig. 31. The reflection operators are:

$$a = \frac{R - z}{R + z} = \text{reflection operator at } R$$

$$b = \frac{1 - zCp}{1 + zCp} = \frac{\beta - p}{\beta + p} = \text{reflection operator at } C \qquad (94)$$

Assuming the capacitor to be initially charged to a voltage E, the initial wave is

$$e = E\varepsilon^{-t/zC} = E\varepsilon^{-\beta t} \tag{95}$$

The $(n + 1)$th reflection at R gives

$$\begin{aligned} e_{(n+1)} &= (1 + a)\, a^n b^n e\, \mathbf{1} \\ &= E\,(1 + a)\, a^n \left(\frac{\beta - p}{\beta + p}\right)^n \varepsilon^{-\beta t}\, \mathbf{1} \end{aligned}$$

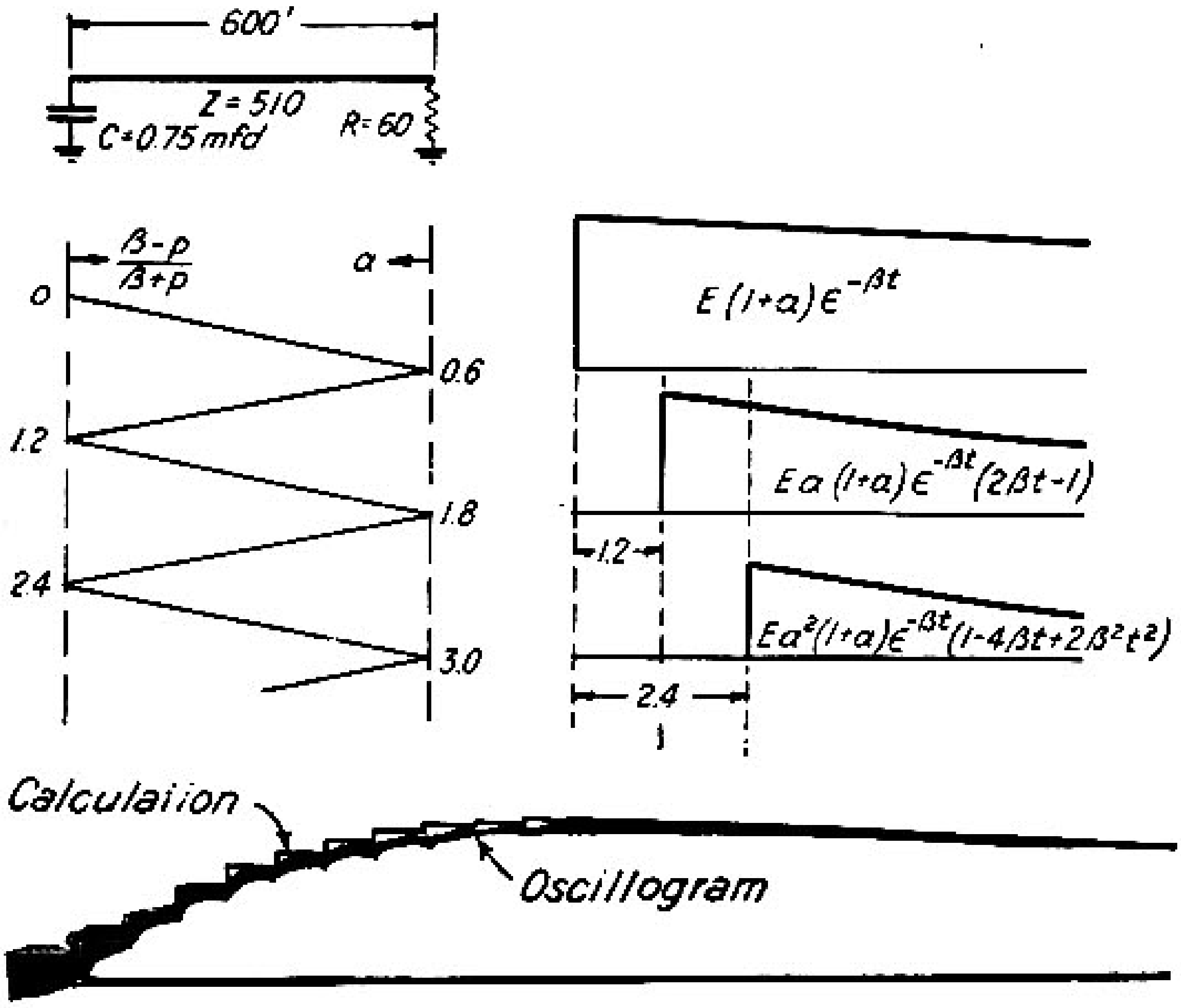


FIG. 31.—Successive Reflections between a Resistor and a Capacitor

(and by the shifting theorem of the Appendix)

$$\begin{aligned} &= E\,(1 + a)\, a^n \varepsilon^{-\beta t}\,(2\,\beta - p)^n \frac{\mathbf{1}}{p^n} \\ &= E\,(1 + a)\, a^n \varepsilon^{-\beta t}\,(2\,\beta - p)^n \frac{t_n}{\underline{|n}} \end{aligned}$$

(and expanding by the binomial theorem)

$$= E\,(1 + a)\, a^n \varepsilon^{-\beta t} \sum_{k=0}^{n} \frac{(-1)^k}{\underline{|k}} \left(\frac{\underline{|n}}{\underline{|n - k}}\right) (2\,\beta)^{n-k} p^k \frac{t^n}{\underline{|n}} \tag{96}$$

But

$$p^k t^n = [n\,(n - 1) \ldots (n - k + 1)]\, t^{(n-k)} = \frac{\underline{|n}}{\underline{|n - k}}\, t^{(n-k)} \tag{97}$$

Therefore

$$e_{(n+1)} = E\,(1 + a)\frac{a^n}{\underline{|n}}\,\varepsilon^{-\beta t} \sum_0^n \frac{(-1)^k}{\underline{|k}} \left(\frac{\underline{|n}}{\underline{|n-k}}\right)^2 (2\,\beta t)^{(n-k)} \tag{98}$$

The total voltage at R at any reflection $(n + 1)$ then is

$$E_{(n+1)} = e_1\,(t) + e_2\,(t - \tau) + e_3\,(t - 2\tau) + \ldots + e_{(n+1)}\,(t - n\tau) \tag{99}$$

where $(t - k\tau)$ denotes that the function starts at $t = k\tau$, counting from the instant of arrival of the first wave at R.

If $R = \infty$, then $a = +1$ and Equation (98) reduces to those of the previous section.

Equation (98) has been plotted in Fig. 31 for the case of a 0.75-microfarad capacitor charging an overhead line 600 ft. long grounded through a resistance of 60 ohms. At the right of the lattice the first three component waves have been plotted, and the accompanying oscillogram shows how closely the calculations check the test results. The successive reflections wipe out the effect of the surge impedance of the transmission line, and thereafter the voltage across the resistor decays according to the law

$$e = E\varepsilon^{-t/RC} \tag{100}$$

Effect of Short Lengths of Cable.*—It was at one time thought that a short length of cable as an entrance into a station for a transmission line was very effective in reducing the surge that could be impressed on the station apparatus, because the surge impedance of a cable is only about a tenth that of an overhead line. But in spite of this great difference in surge impedance, the voltage at the end of the cable rapidly builds up by successive reflections if the incident wave is long compared to the length of the cable; and about the only effect of the cable is to slow down the wave front. As an example of how the effect of short lengths of cable can be calculated by means of the lattice, there are shown in Fig. 32 two short lengths (600 ft.) of cable separated by a short length (500 ft.) of overhead line. The surge impedance of the line is taken as $Z_1 = 500$, and that of the cable as $Z_2 = 50$. The velocities of propagation for the line and cable are 1000 ft. per ms. and 600 ft. per ms. respectively, so that on a time axis

* "Study of the Effect of Short Lengths of Cable on Traveling Waves," by K. B. McEachron, T. G. Hemstreet, and H. P. Seelye, *A.I.E.E. Trans.*, Vol. 49, p. 1432.

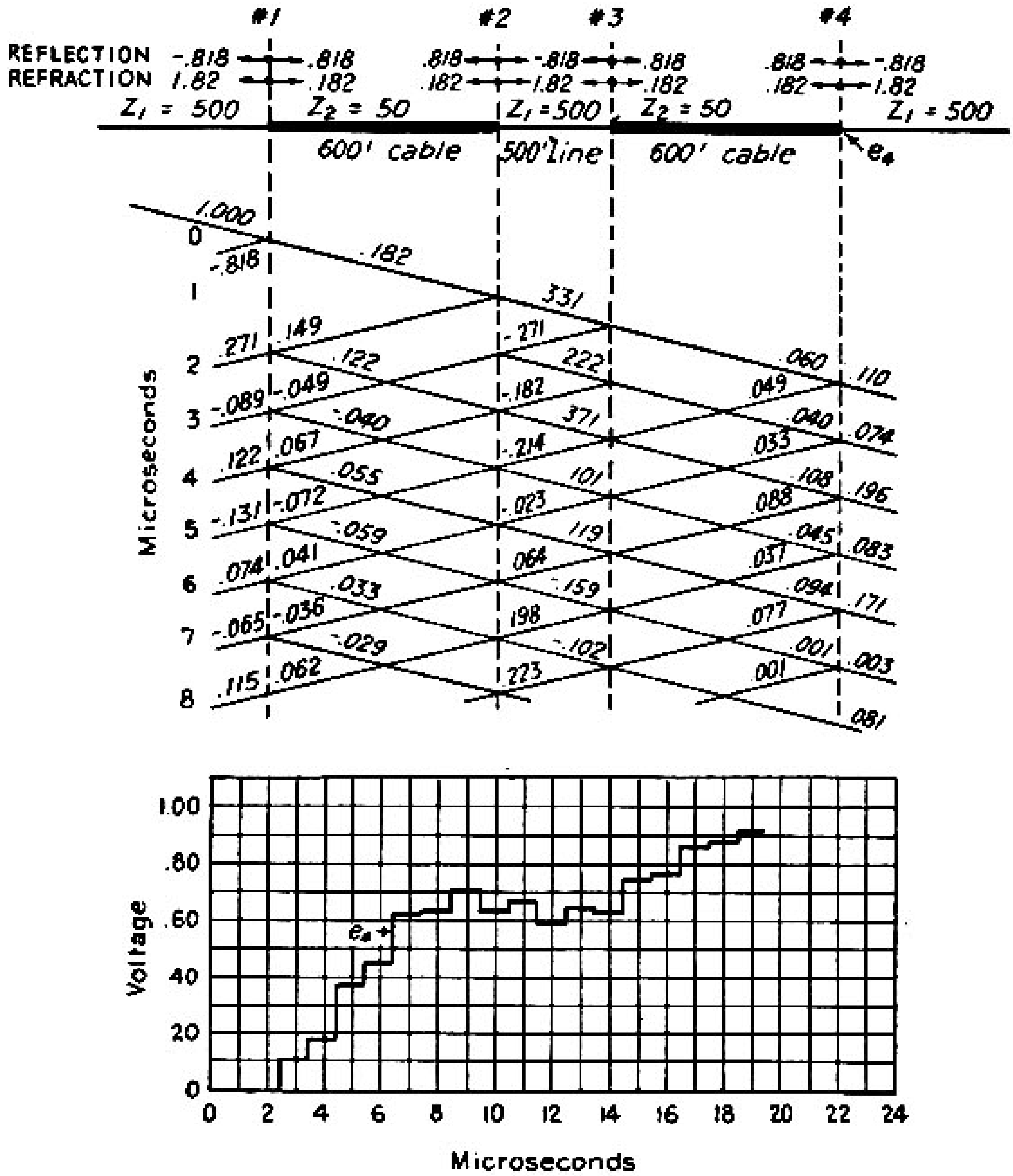


FIG. 32.—Effect of Short Lengths of Cable

the cable lengths are one microsecond long. The reflection and refraction operators are:

$$a_1 = -a_2 = a_3 = -a_4 = a_4' = -a_3' = a_2' = -a_1' = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -\ 0.818$$

$$b_1 = b_3 = b_4' = b_2' = \frac{2\, Z_2}{Z_2 + Z_1} = 0.182$$

$$b_2 = b_4 = b_3' = b_1' = \frac{2\, Z_1}{Z_2 + Z_1} = 1.82$$

In this example, the numerical values of the operators have been entered directly on the lattice in order to facilitate the calculations. This procedure is always preferable when dealing with pure resistance or surge impedance junctions. Attenuation has been neglected in this calculation, because the purpose of the analysis is to show the effect of the two different surge impedances on the transmitted wave when successive reflections are taken into account. The short lengths of cable do not prevent the transmitted wave from ultimately reaching 100 per cent of the value of the incident wave if the incident wave is sufficiently long. However, the effective wave front of the transmitted wave is lengthened considerably, which represents a real advantage as far as the stresses on machine windings are concerned.

SUMMARY OF CHAPTER IV

Successive (or repeated) reflections may be kept track of by means of a lattice, and therefrom the equations for any junction may be written. This lattice gives the position, direction, shape, and previous history of every wave at all instants of time. Attenuation and distortion may be included, if their defining functions are known. By means of this lattice such problems as the charging of a line from a d-c. generator or an impulse generator, effect of short lengths of cable, ground-wire calculations, etc., may be easily computed. When the incident wave is a simple exponential, or compounded of exponentials, the solutions for successive reflections can usually be found, but the equations are laborious and awkward (often leading to double and triple summations in series). Fortunately, however, in many engineering applications, only the first few terms of these multiple series are of importance. For example, ordinarily not more than three or four terms are required to determine the maximum voltage due to repeated reflection in the circuit of Fig. 38. Other examples of successive reflections will be found in Chapters X and XV.

CHAPTER V

SOME PROTECTIVE SCHEMES

The oldest method of protection against the harmful effects of high-voltage traveling waves is a simple gap set to spark over at a voltage below the impulse strength of the apparatus to be protected. The gap is used in modern practice to establish the voltage level in schemes for the coordination of system insulation, in which the insulation links are graded so that the less essential and most accessible parts protect the more vital components. An important link in the coordination scheme is the lightning arrester, whose function is to limit the impulse voltage to values such that the gap does not flash over. If the gap is allowed to flash over, the power frequency follow-up current will not be interrupted and discontinuity of service will result. The impulse sparkover characteristics of insulator strings, oil circuit-breaker and transformer bushings, bus insulators, gaps, etc., are usually quite different and sensitive to small changes; that is, even for the same wave the sparkover voltage varies as much as ± 10 per cent and depends also upon the polarity of the wave. Therefore, in order properly to coordinate the system for all waves and conditions, it would be necessary to carry the idea far beyond its economic limit. However, a compromise can be worked out and the plan used as an effective second line of defense in case the lightning arresters fail to work. In this connection it becomes necessary to inquire into the sparkover characteristics of the different insulators and gaps.

The sphere gap is practically instantaneous and will spark over without appreciable time lag and at consistent values for given atmospheric conditions, provided that the spheres are clean and polished, that their spacing is well within their range of precision, and that they are sufficiently removed from the influence of foreign equipotential surfaces and fields, such as adjacent grounds. But although the sphere gap is practically instantaneous, it is not absolutely so, and oftentimes fails to record extremely high-frequency oscillations superimposed on the crest of an impulse. No definite data are available on the time lag of sphere gaps of various sizes and spacings; but the time lag of all other gaps and of insulation failure is so great compared with that of the sphere gap that the latter establishes the reference level.

Sparkover Characteristics of Insulators and Gaps.*—The sparkover characteristics of needle or point gaps, insulator strings, bushings, and other insulators, all follow a law which can be empirically expressed by

$$e = e_0\left(1 + \frac{a}{\sqrt{t}}\right) \quad \text{or} \quad \beta = \frac{e}{e_0} = \left(1 + \frac{a}{\sqrt{t}}\right)$$

in which e_0 is the ultimate d-c. sparkover, t is the time lag, and a an empirical constant depending upon the type of gap, shape and polarity of the wave, temperature, barometric pressure, and humidity.

It is an interesting conjecture that sparkover depends upon the the corona energy absorbed by the gap. Assuming that the mechanism of sparkover is a progressive corona, and the minimum d-c. voltage at which such a corona can form is e_0—the ultimate d-c. sparkover voltage—the corona energy absorbed by the gap, according to the quadratic law, is

$$W = \int_{t_0}^{t} (e - e_0)^2\, dt$$

or

$$a^2 = \frac{W}{{e_0}^2} = \int_{t_0}^{t}\left(\frac{e}{e_0} - 1\right)^2 dt = \int_{t_0}^{t} (\beta - 1)^2\, dt \tag{101}$$

where $e = f(t)$ = applied impulse as function of time.
t_0 = instant at which $e = f(t_0) = e_0$.
t = instant of sparkover.

Integrating (101) for any specific case there results an equation of the form

$$a^2 = \int_{t_0}^{t} (\beta - 1)^2\, dt = F(t) - F(t_0) \tag{102}$$

from which the instant of sparkover, t, may be found and therefrom the sparkover voltage $e = f(t)$, assuming that W is fixed for a given gap and a given mode of application of voltage, that is, a given wave shape. If (102) integrates into a transcendental equation, or if $f(t)$

* "The Effect of Transient Voltages on Dielectrics. IV," by F. W. Peek, Jr., *A.I.E.E. Trans.*, Vol. 49, p. 1456.

can not be expressed analytically, then the following graphical method of Fig. 33 is applicable:

1. Draw the curve of the applied impulse $e = f(t)$ and the e_0 line intersecting $e = f(t)$ at $t = t_0$ and $t = t_1$ on the front and tail of the wave respectively.
2. Construct the curve of $(\beta - 1)^2$ between the limits t_0 and t_1.
3. By trial find the area A under the $(\beta - 1)^2$ curve out to a time t' (where $t_0 \angle t' \angle t_1$) such that $a^2 = A$. Then t' is the instant of sparkover and $e = f(t')$ is the sparkover voltage.

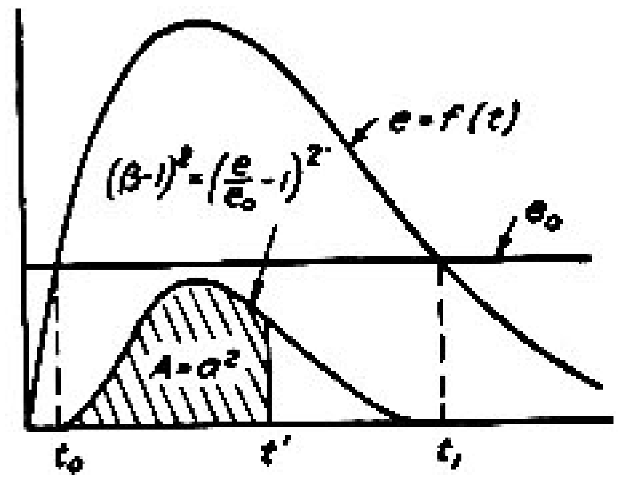


FIG. 33.—Graphical Determination of Sparkover Characteristic

A few examples will make the application of (102) clear.

I. *Sparkover on a Uniformly Rising Front*, Fig. 34.

$$e = f(t) = e_0\, t\ t_0 = \text{applied impulse.}$$

$$\beta = e\ e_0 = t\ t_0 = \text{impulse ratio.}$$

$$a^2 = \int_{t_0}^{t} \left(\frac{t}{t_0} - 1\right)^2 dt = \frac{t_0}{3}(\beta^3 - 3\beta^2 + 3\beta - 1).$$

$$\frac{(\beta - 1)^3}{\beta} = \frac{3\, a^2}{\beta t_0} = \frac{3\, a^2}{t} \tag{103}$$

II. *Sparkover on Top of a Rectangular Wave*, Fig. 34.

$$e = E = \text{applied impulse.}$$

$$t_0 = 0.$$

$$\beta = E/e_0 = \text{impulse ratio.}$$

$$a^2 = \int_0^t (\beta - 1)^2\, dt = (\beta - 1)^2\, t.$$

Therefore

$$\beta = 1 + \frac{a}{\sqrt{t}} \tag{104}$$

III. *Sparkover on Tail of an Exponential Wave*, Fig. 34.

$$e = E\varepsilon^{-\alpha t} = \text{applied impulse.}$$

$$t_0 = 0.$$

$$\beta = E/e_0 = \text{impulse ratio.}$$

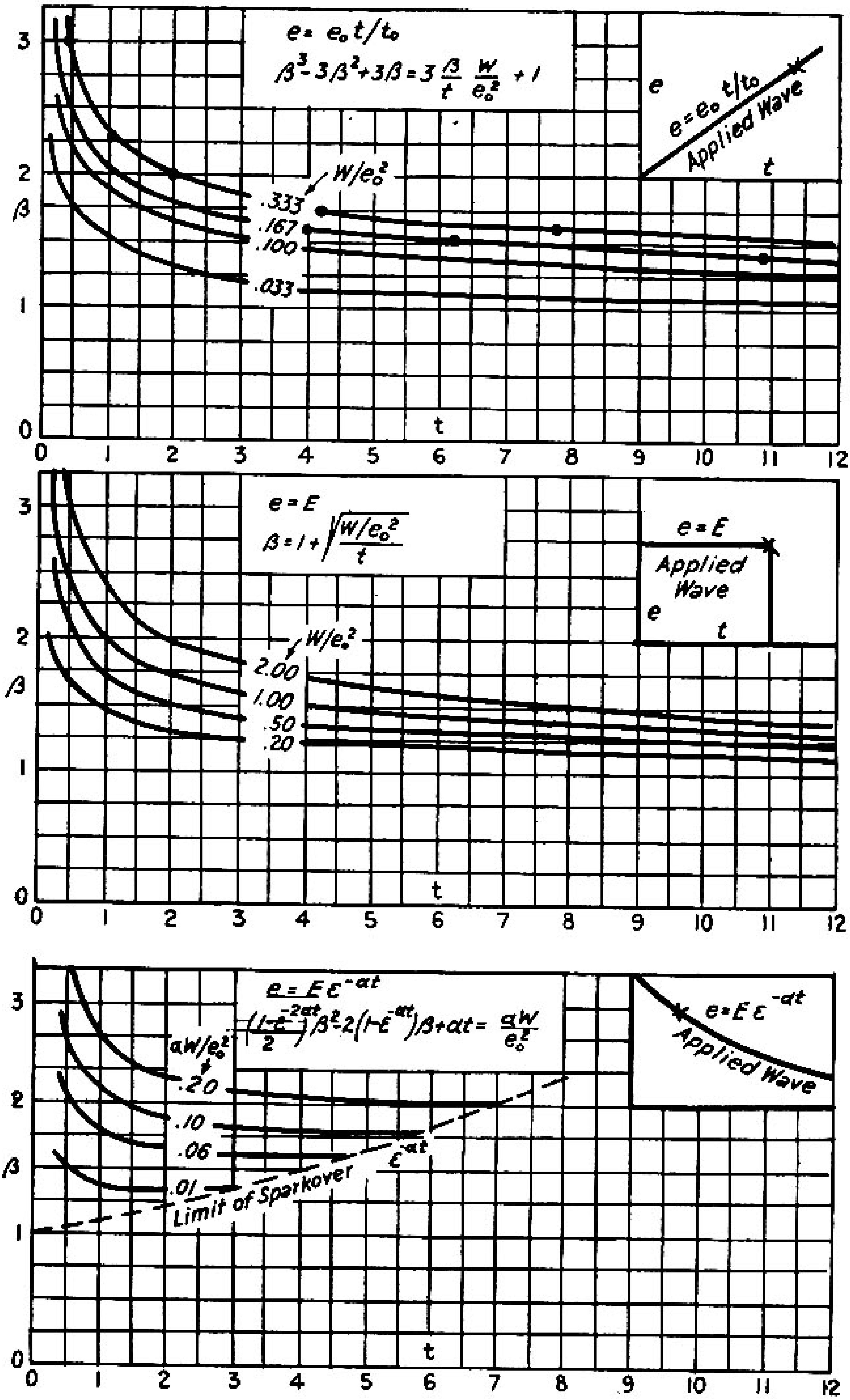


FIG. 34.—Sparkover Characteristics for Different Wave Shapes

$$a^2 = \int_0^t \left(\frac{E}{e_0}\varepsilon^{-\alpha t} - 1\right)^2 dt$$

$$= \left(\frac{1 - \varepsilon^{-2\alpha t}}{2}\right)\frac{\beta^2}{\alpha} - 2\,(1 - \varepsilon^{-\alpha t})\,\frac{\beta}{\alpha} + \frac{\alpha t}{\alpha} \tag{105}$$

The time t_1 beyond which sparkover can not occur is

$$e_0 = E\varepsilon^{-\alpha t_1} \quad \text{or} \quad t_1 = \frac{\log \beta}{\alpha} \tag{106}$$

The above three cases have been plotted in Fig. 34. The most surprising point brought out by a comparison of these curves with actual sparkover data is the considerable difference in the energy that it takes to break down a given gap with different types of waves. It appears that much less energy is required to cause sparkover on a rising front than on the crest or tail of a wave, and therefore suggests that the rate of application of voltage has considerable effect on the sparkover characteristics.

It is theoretically possible, on the basis of the above analysis, to predict the complete sparkover curve if the impulse ratio at the 50 per cent sparking point is known. If the impulse ratio where the tail of the applied wave crosses the e_0 line at $t = t_1$ is β_1, then, since the sparkover characteristic follows the empirical law

$$\beta = 1 + \frac{a}{\sqrt{t}}$$

it follows that

$$a = (\beta_1 - 1)\sqrt{t_1}$$

where t_1 corresponds to $e_0 = \beta_1 E$. Therefore

$$\beta = 1 + (\beta_1 - 1)\sqrt{\frac{t_1}{t}} \tag{107}$$

Thus by taking sphere gap measurements at the 50 per cent sparking point (which occurs approximately at $t = t_1$), it is theoretically possible to calculate the complete sparkover characteristic. However, sparkover is quite erratic under even the best of controlled conditions, and the variation is especially great on the falling tail of the wave, so that no great amount of reliance can be placed on a curve calculated from the 50 per cent sparkover data.

Coordination of Insulation.—Fig. 35*a* illustrates the ideal scheme of insulation coordination, in which each successive link protects the preceding link throughout the entire sparkover range. Owing to the difference in sparkover characteristics of different insulators, this ideal situation can not be entirely realized.

Lightning Arresters.—The simplest form of surge limiting device is a sparkgap of some kind or other—spheres, points, rods, and horns are all in common use. But, as has been previously pointed out, a gap by itself is unable to interrupt the system frequency power arc, and consequently may involve a circuit-breaker operation to clear the fault, with the resulting discontinuity of service. The lightning arrester was therefore introduced to circumvent this defect of the simple gap.

One of the best lightning-arrester materials so far developed is

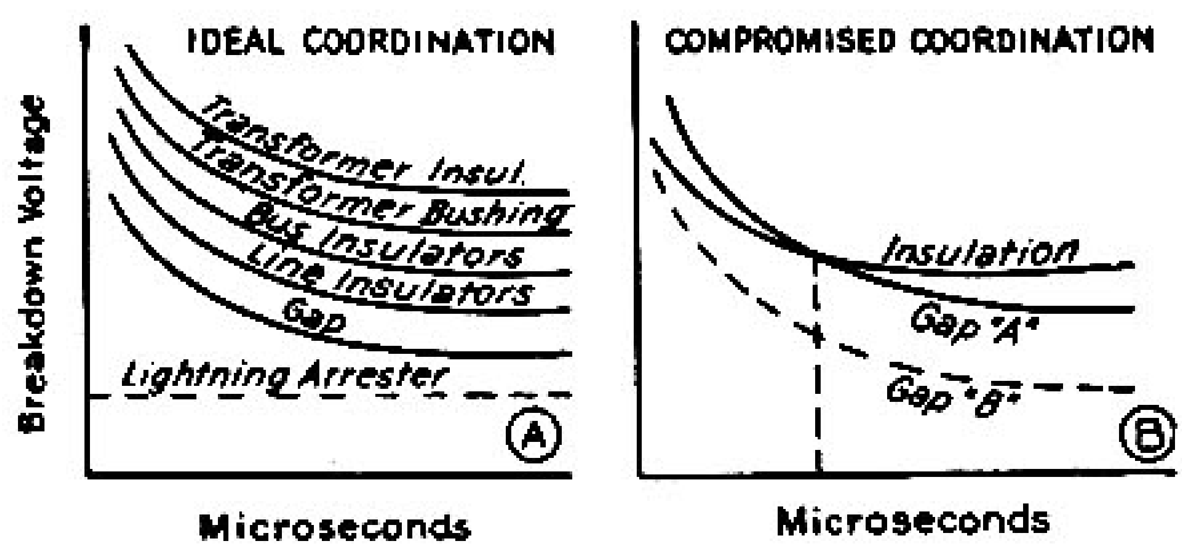


FIG. 35.—Sparkover Characteristics and Insulation Coordination

Thyrite.* Thyrite is characterized by its volt-ampere characteristic,

$$E\, I^{(a-1)} = R\, I^a = C \tag{108}$$

in which a is the " exponent " and C is the " constant " whose value is dependent on the amount of material used. Taking the logarithm of both sides of this equation there is

$$\log E + (a - 1) \log I = \log R + a \cdot \log I = \log C \tag{108a}$$

Thus the equation is a straight line if plotted on log-log paper, Fig. 36. If $a = 1$, then $E = C$ for all values of I, and therefore the voltage across the arrester can not exceed the limiting value $E = C$. This is the ideal sought by designers of lightning arresters.

The surge impedance of the transmission line plays an essential part in the functioning of the lightning arrester. Referring to Fig. 37,

* " Thyrite—A New Material for Lightning Arresters," by K. B. McEachron, *A.I.E.E. Trans.*, Vol. 50, 1930.

which represents a Thyrite arrester at the end of a transmission line of surge impedance Z, the equation at the arrester is

$$2\,e = e_R + Z\,i \tag{109}$$

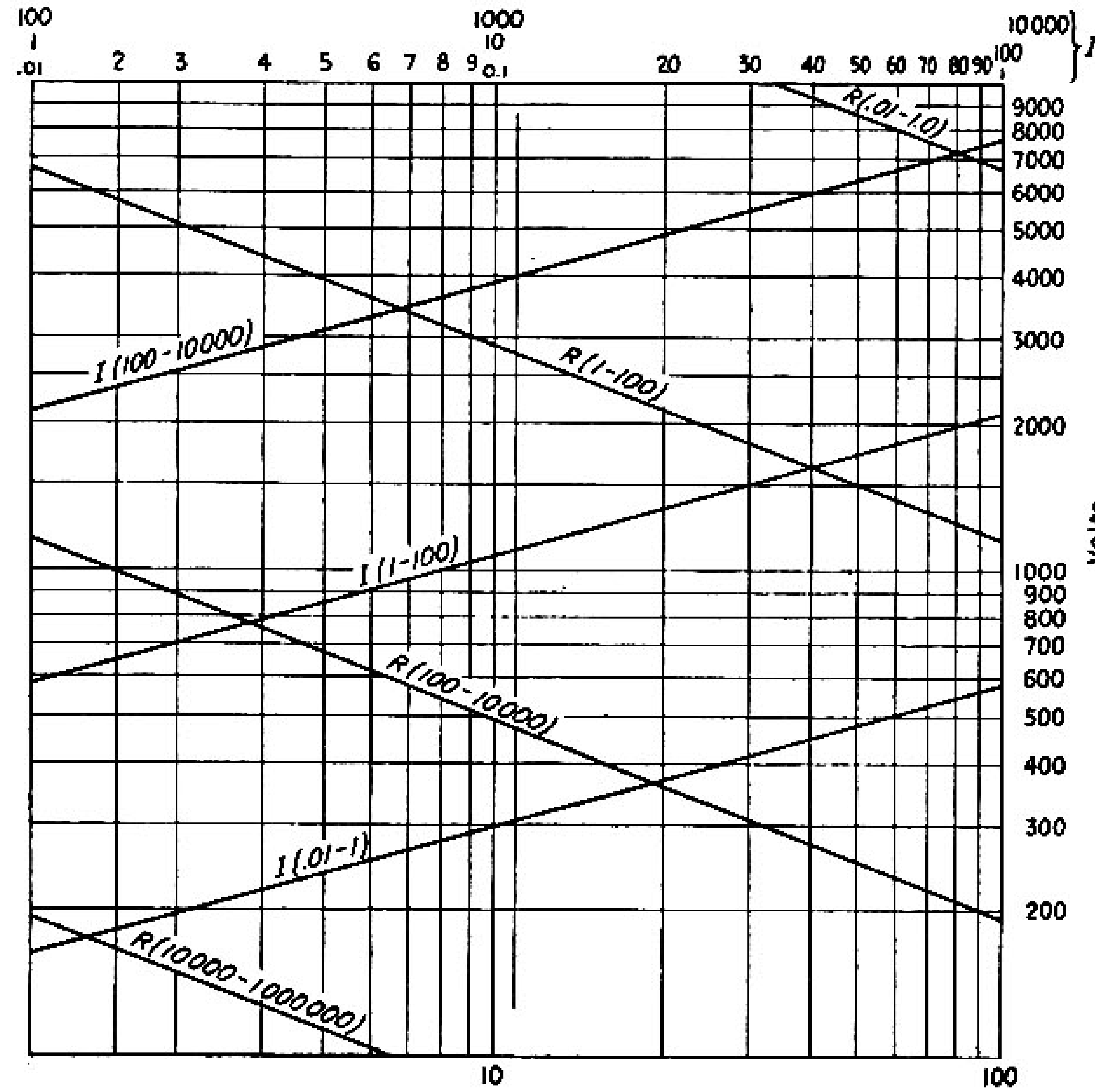


FIG. 36.—Thyrite Characteristics

where e is the free traveling wave, e_R the voltage across the arrester, and i the arrester current. This equation may be solved graphically

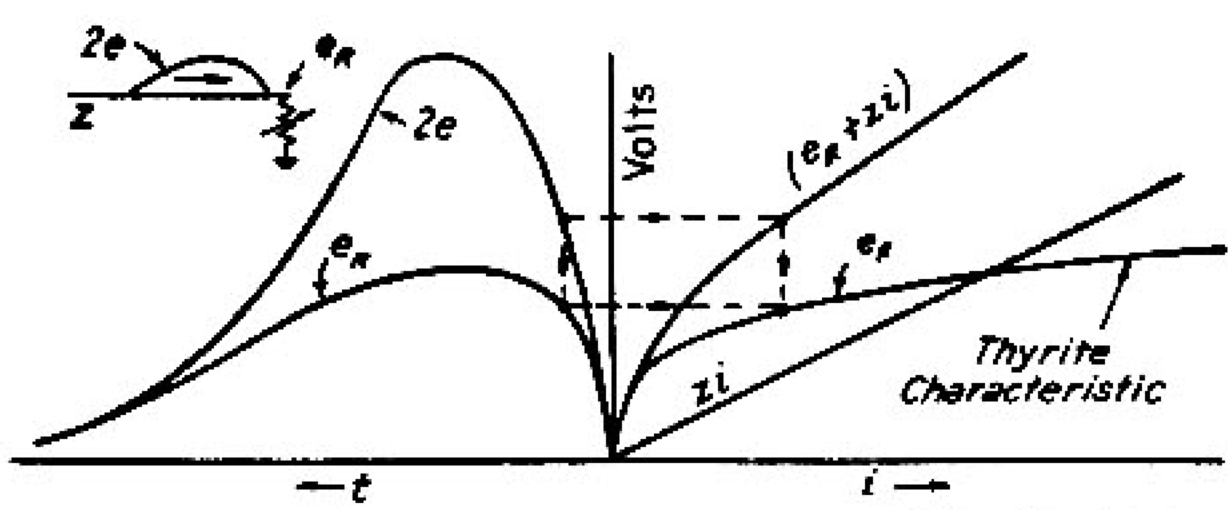


FIG. 37.—Graphical Determination of Voltage Across a Thyrite Lightning Arrester

as shown in Fig. 37, in which $2\,e$ is plotted against t, and $(e_R + Z\,i)$ is plotted against i. Then for any i there is a certain e_R and a corresponding t from the $2\,e$ curve, so that e_R may be plotted directly against t. It is usually quicker to solve the equation by tabular method as follows:

i	$Z\,i$	e_R	$2\,e = Z\,i + e_R$	t
(1)	(2)	(3)	(4)	(5)

For any i find $Z\,i$ and e_R and their sum, which must be equal to $2\,e$ and therefore defines t on the $e_R \sim t$ curve.

For a Thyrite arrester at the junction between two surge impe-

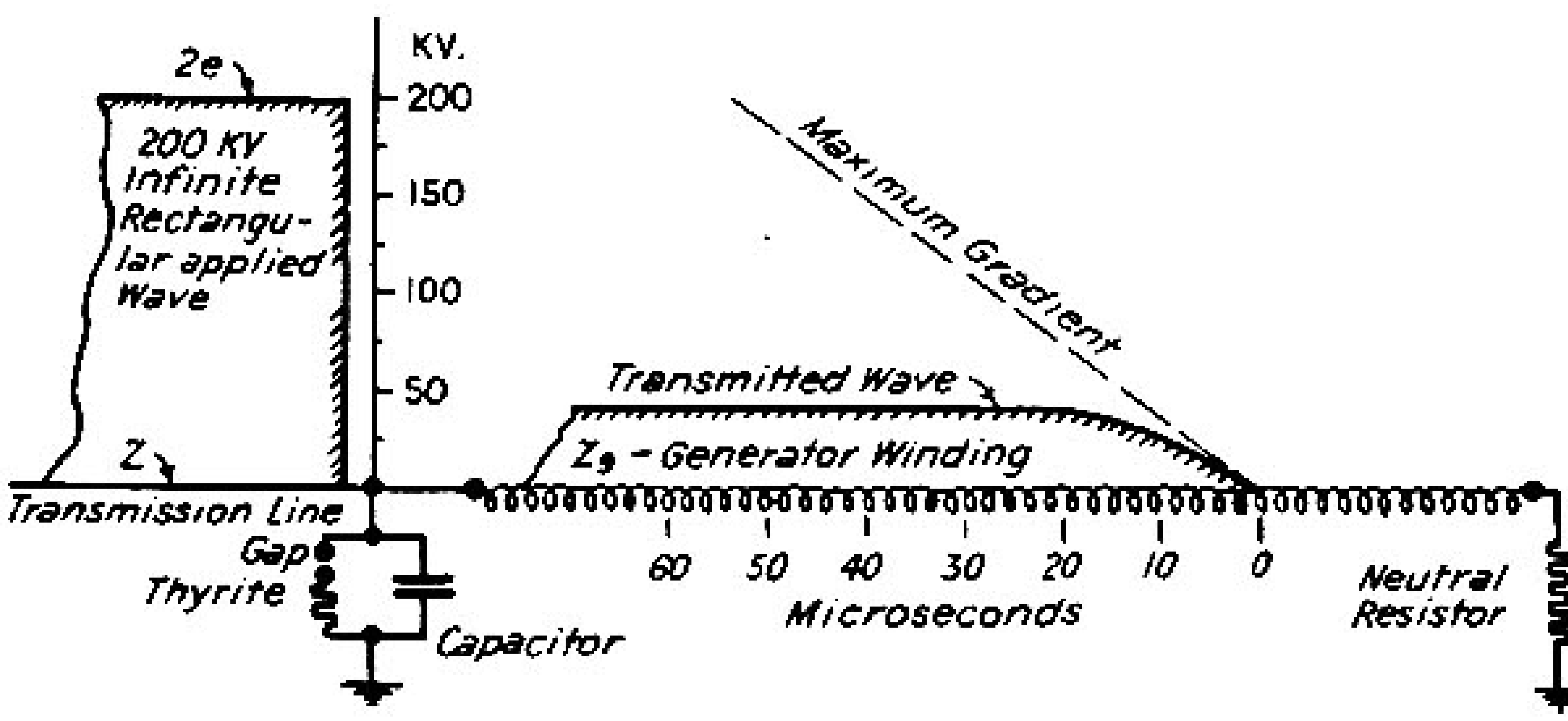


FIG. 38.—Protection of Generator Windings

dances Z_1 and Z_2, for example an overhead line and cable junction, the equation to be satisfied is

$$2\,e = e_R + Z_1\,i = e_R + Z_1\left(i_R + \frac{e_R}{Z_2}\right) = \left(1 + \frac{Z_1}{Z_2}\right)e_R + Z_1\,i_R \quad (109a)$$

This equation may be solved either by the graphical or tabular method in the same way as described for a single line terminating at the arrester.

Protection of Rotating Apparatus.*—A scheme that has been proposed for the protection of rotating machines which are directly connected to overhead lines is illustrated in Fig. 38. The winding of a synchronous generator acts like a surge impedance Z_2 of 600 to 1200

* "Voltage Oscillations in Armature Windings under Lightning Impulses," by E. W. Boehne, *A.I.E.E. Trans.*, Vol. 49, 1930.

ohms, shunted by a small terminal capacitance. An average figure for the velocity of propagation of waves in armature windings is 55 ft. per ms., thus only about one-twentieth that of waves on an overhead transmission line. The gradient, or turn-to-turn stress, in generator windings, is directly proportional to the steepness of the wave front, for the steeper the wave front, the greater the voltage across a given length of the winding. In order to reduce the steepness of the front a large terminal capacitance C is connected to ground at the line end. A capacitor by itself, however, will not limit the ultimate voltage for sufficiently long applied waves; and it is therefore necessary to have in parallel therewith a lightning arrester. If the neutral end of the generator winding is isolated, reflections take place, and the internal voltages to ground are doubled in value. These reflections may be eliminated by connecting a resistor R equal to or less than the surge impedance of the generator winding between the neutral and ground. Of course, any resistance less than Z_2 is desirable from this standpoint, because the reflections therefrom are then waves of reduction. The equation to be satisfied at the line terminal is:

$$2\,e = e_R + Z_1\,i_1 = e_R + Z_1\left(i_R + \frac{e_R}{Z_2} + C\frac{de_R}{dt}\right) \qquad (110)$$

This equation may be solved through a step-by-step process, upon replacing the differentials by increments and rearranging the equation in the form

$$\Delta\,e_R = \left[\frac{2\,e - Z_1\,i_R - (1 + Z_1/Z_2)\,e_R}{Z_1\,C}\right]\Delta t \qquad (111)$$

where $2\,e$, i_R, and e_R are the average values over the interval Δt. As an example of its application, take

$Z_1 = 500.$
$Z_2 = 1000.$
$C = 0.12$ microfarad.
$2\,e = 200{,}000.$
$N = 15$ Thyrite disks $a = 0.72$, $C = 580$, Fig. 36.

Now fill in the following table, assuming trial values of i_R for each incremental step, until one is found such that the corresponding value of e_R is equal to the previous value of e_R plus the calculated increment Δe_R. The method is greatly facilitated by plotting the $Z_1\,i_R$ and e_R curves as the calculation progresses and therefrom extrapolating ahead over each internal Δt in order to estimate more closely the next value of i_R to assume.

Δt	t	$2e$	i_R	$Z_1 i_R$	e_R	Δe_R
	0	200	0	0	0	
5						16
	5	200	9	4.5	16	
5						13
	10	200	76	38	29	
5						8.5
	15	200	185	92	37.5	
5						4.0
	20	200	270	135	41.5	
10						0.0
	30	200	270	135	41.5	

The effect of the capacitance has been to lengthen the wave front some 20 ms., while the Thyrite lightning arrester holds the voltage down to 45 kv. as compared with 200 kv. without the arrester. In the example given, it has been assumed, for simplicity, that the Thyrite is in the circuit all the time. Actually, there would be a gap in the lightning-arrester circuit, and the Thyrite would not come into play until a predetermined voltage was reached, but during this period the capacitor would be effective in reducing the wave front.

Choke Coils.—The use of choke coils as an adjunct to lightning arresters has been discontinued. However, occasionally choke coils are used on transmission lines for other purposes. When so used the possibility of high-frequency oscillations must be taken into account. Fig. 39 represents an inductance L (for example, a choke coil) in series with a capacitance C (representing the effective capacitance of the bus, transformers, oil circuit-breakers, etc.), a small damping resistance R, and the surge impedance Z. The system is protected by a gap G, which in one case is assumed not to spark over and in the other case to spark over in 4 ms. Calculations are made for an infinite rectangular wave on the right, and for a 7/20-ms. wave on the left. It is evident that the slower wave front does not cause as high oscillations as the abrupt wave front. In the event of a sparkover, R, L, and C comprise a local oscillating circuit, and if the natural frequency of internal oscillations of any connected apparatus, such as a transformer, is in the neighborhood of this frequency, there is danger from severe resonance. If there were no reactance in the circuit the voltage across the capacitor could do no more than double, as compared with 270 per cent for the abrupt front wave, and there would be no oscillations.

Current Limiting Reactors.—Current limiting reactors are used to limit the short-circuit current of important feeders and generators, and as their inductance may be several hundred times that of the con-

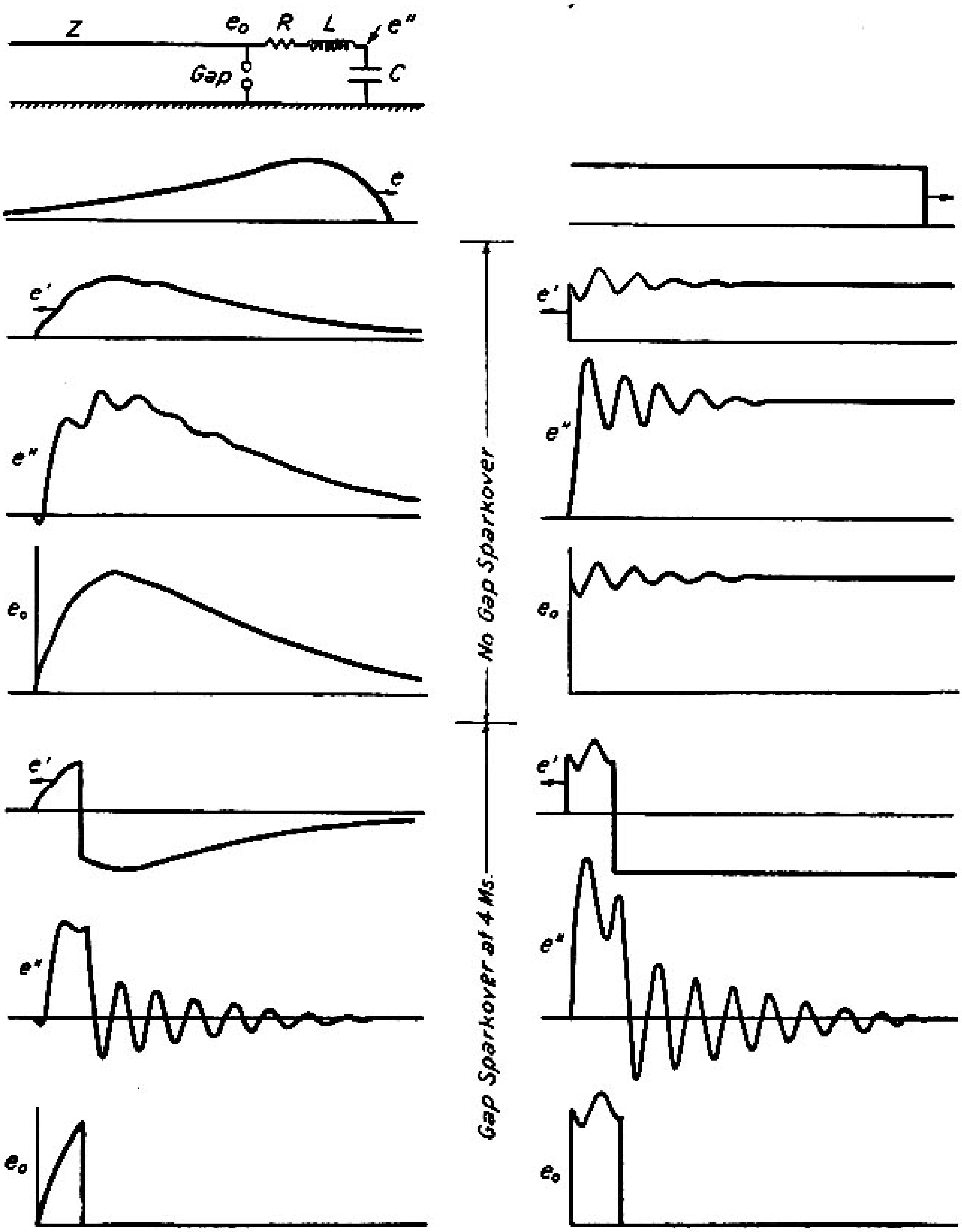


FIG. 39.—Oscillations Caused by a Choke Coil in Series with a Transformer

ventional choke coil, they may have a decided influence on lightning waves and surges. A typical installation of such reactors is indicated diagrammatically in Fig. 40. In order to prevent the reactor from entering into oscillation with the capacitance of the terminal apparatus, and thereby building up excessive voltages, it is advisable to shunt the reactor with a resistor,* as illustrated in Fig. 41, where C is the total capacitance of the terminal equipment (bus, transformers, generators, etc.), and Z_2 is the net surge impedance of all outgoing feeders and machine windings. To complete the picture there should be an inductance in shunt with C, to represent the inductance of the windings, because the equivalent circuit of grounded neutral transformers and windings is a large inductance in parallel with an effective terminal capacitance. But the inclusion of this inductance in the analysis only unnecessarily complicates the mathematics, and adds nothing essential to the character of the oscillation, because, at the high frequencies with which we are concerned, the inductance acts as an open circuit. The small series resistance r shown in Fig. 41 represents the reactor resistance as modified by skin effect.

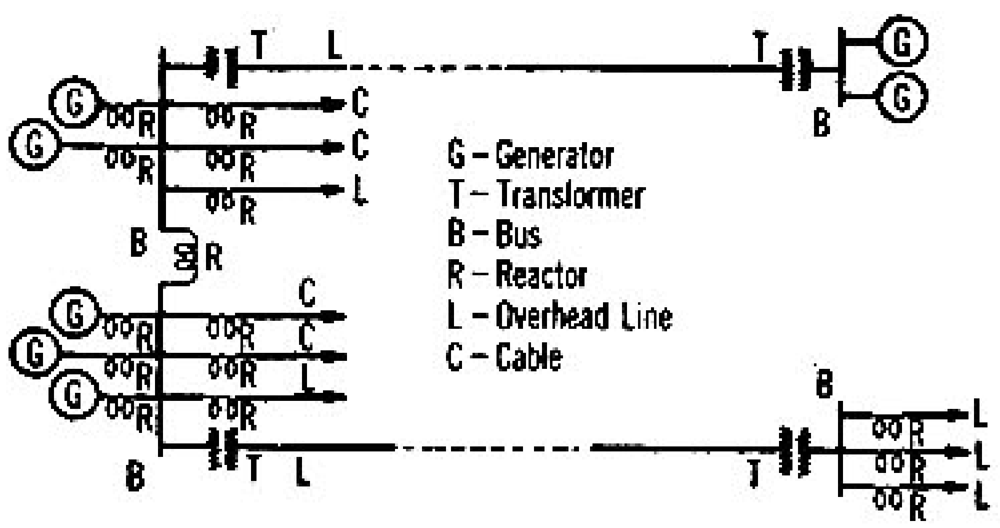


FIG. 40.—Single Line Diagram of a Typical System Using Current Limiting Reactors

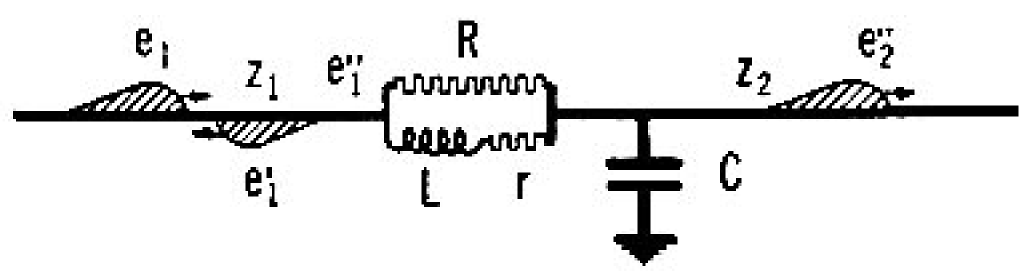


FIG. 41.—Equivalent Circuit of Lines, Reactor and Terminal Apparatus

The generalized terminal impedance at the junction is

$$Z_0\,(p) = Z_1\,(p) + Z\,(p) = \frac{R\,(r + L\,p)}{r + R + p L} + \frac{z_2}{1 + z_2\,C\,p}$$

The reflection operator is

$$\frac{Z_0\,(p) - z_1}{Z_0\,(p) + z_1} = \frac{(R - z_1)}{(R + z_1)}\,\frac{p^2 + A\,p + B}{p^2 + 2\,\alpha\,p + \omega_0^2}$$

The refraction operator is

$$\frac{2\,Z\,(p)}{Z_0\,(p) + z_1} = \frac{2}{C\,(R + z_1)}\,\frac{p + F}{p^2 + 2\,\alpha\,p + \omega_0^2}$$

* "Shunt Resistors for Reactors," by F. H. Kierstead, H. L. Rorden, and L. V. Bewley, *A.I.E.E. Trans.*, Vol. 49.

where

$$\alpha = \frac{(R + z_1 + z_2)\,L + (r\,R + rz_1 + R\,z_1)\,z_2\,C}{2\,(R + z_1)\,z_2\,L\,C}$$

$$\omega_0^2 = \frac{(z_1 + z_2)\,(r + R) + r\,R'}{(R + z_1)\,z_2\,L\,C}$$

$$\omega^2 = \omega_0^2 - \alpha^2 = -\,\Omega^2$$

$$A = \frac{(R - z_1 + z_2)\,L + (r\,R - rz_1 - R\,z_1)\,z_2\,C}{(R - z_1)\,z_2\,L\,C}$$

$$B = \frac{(z_2 - z_1)\,(r + R) + r\,R}{(R - z_1)\,z_2\,L\,C}$$

$$F = \frac{r + R}{L}$$

Let the incident wave be given by

$$e_1 = E_1\varepsilon^{-at}$$

where time is counted from the instant when the wave arrives at the junction. Applying the *reflection* and *refraction* operators and using Heaviside's shifting theorem, there is, after some simplification,

$$\begin{aligned} e_1' &= x\,E_1\varepsilon^{-at} + y\,E_1\varepsilon^{-\alpha t}\sin\,(\omega t + \phi) \\ &= x\,E_1\varepsilon^{-at} + E_1\varepsilon^{-\alpha t}\,(W_1\varepsilon^{\Omega t} - W_2\varepsilon^{-\Omega t}) \qquad (112) \\ &= \text{reflected wave} \end{aligned}$$

$$\begin{aligned} e_2'' &= u\,E_1\varepsilon^{-at} + v\,E_1\varepsilon^{-\alpha t}\sin\,(\omega t + \psi) \\ &= u\,E_1\varepsilon^{-at} + E_1\varepsilon^{-\alpha t}\,(J_1\varepsilon^{\Omega t} - J_2\varepsilon^{-\Omega t}) \qquad (113) \\ &= \text{refracted wave} \end{aligned}$$

where

$$x = \frac{R - z_1}{R + z_1}\,\frac{a^2 - a\,A + B}{\omega_0^2 - 2\,\alpha\,a + a^2}.$$

$$y = \frac{R - z_1}{R + z_1}\,\frac{\sqrt{m^2 + n^2}}{(\omega_0^2 - 2\,\alpha\,a + a^2)\,\omega}.$$

$$\phi = \tan^{-1}\left(\frac{n}{m}\right).$$

$$m = [(A - a - \alpha)\,(\omega_0^2 - 2\,\alpha\,a + a^2) \\ -(a^2 - a\,A + B)\,(\alpha - a)].$$

$$n = \omega\,(\omega_0^2 - 2\,\alpha\,a + a\,A - B).$$

$$u = \frac{2}{C(R + z_1)} \frac{(F - a)}{(\omega_0^2 - 2\alpha a + a^2)}.$$

$$v = \frac{2}{C(R + z_1)} \frac{\sqrt{k^2 + l^2}}{(\omega_0^2 - 2\alpha a + a^2)\,\omega}.$$

$$\psi = \tan^{-1}\left(\frac{l}{k}\right).$$

$$k = [\omega_0^2 - \alpha a - (\alpha - a) F].$$

$$l = (a - F)\,\omega.$$

$$W_1 = \frac{R - z_1}{R + z_1} \frac{(m - j n)}{2\,\Omega\,(\omega_0^2 - 2\alpha a + a^2)}.$$

$$W_2 = \frac{R - z_1}{R + z_1} \frac{(m + j n)}{2\,\Omega\,(\omega_0^2 - 2\alpha a + a^2)}.$$

$$J_1 = \frac{1}{C(R + z_1)} \frac{(k - j l)}{\Omega\,(\omega_0^2 - 2\alpha a + a^2)}.$$

$$J_2 = \frac{1}{C(R + z_1)} \frac{(k + j l)}{\Omega\,(\omega_0^2 - 2\alpha a + a^2)}.$$

The total voltage at the reactor is

$$e_1'' = e_1 + e_1'$$

From the above equations it is evident that a simple exponential incident wave striking the junction gives rise, in both the reflected and transmitted waves, to an exactly similar wave plus a damped oscillation. This damped oscillation has a frequency and decrement factor depending on every circuit constant present. In case $\alpha^2 > \omega_0^2$, the oscillation ceases to exist, and the oscillatory component degenerates to simple exponential decay.

As a check on the computations it is worth observing, from the previous equations, that (making $t = 0$)

$$x + y \sin\phi = \frac{R - z_1}{R + z_1} \tag{113}$$

$$u + v \sin\psi = 0 \tag{114}$$

These conditions are also evident from physical considerations, for at the first instant the inductance acts as an open circuit and the capacitance as a short circuit to ground. Therefore at this first instant the potential at the capacitance must be zero, and e_1' reflects as from a line grounded through a resistor R.

Owing to their transcendental character, the instantaneous maxima of these equations can not be found except by trial or graphically. However, the maximum amplitudes of the damped oscillations are readily obtained by differentiating with respect to t and equating to zero. These maxima occur at

$$t_1 = \frac{1}{\omega}\left(\tan^{-1}\frac{\omega}{\alpha} - \phi\right) \tag{115}$$

$$t_2 = \frac{1}{\omega}\left(\tan^{-1}\frac{\omega}{\alpha} - \psi\right) \tag{116}$$

and the maxima of the oscillations are

$$y\left(\frac{\omega}{\omega_0}\right) E_1\, \varepsilon^{-\alpha t_1} \tag{117}$$

$$v\left(\frac{\omega}{\omega_0}\right) E_1\, \varepsilon^{-\alpha t_2} \tag{118}$$

Obviously, they can not exceed (yE_1) and (vE_1) respectively, but approach these limiting values as $\alpha \rightarrow 0$.

A large number of specific cases calculated from these equations are given in the A.I.E.E. paper referred to, of which two representative cases are reproduced in Figs. 42 and 43 of this book. The voltages e_1'' and e_2'' on both sides of the reactor, with and without a shunt resistor, are shown. In Fig. 42 the values corresponding to an infinite rectangular incident wave are plotted positive, and those corresponding to a 7/20-ms. wave are plotted negative, to avoid interference.

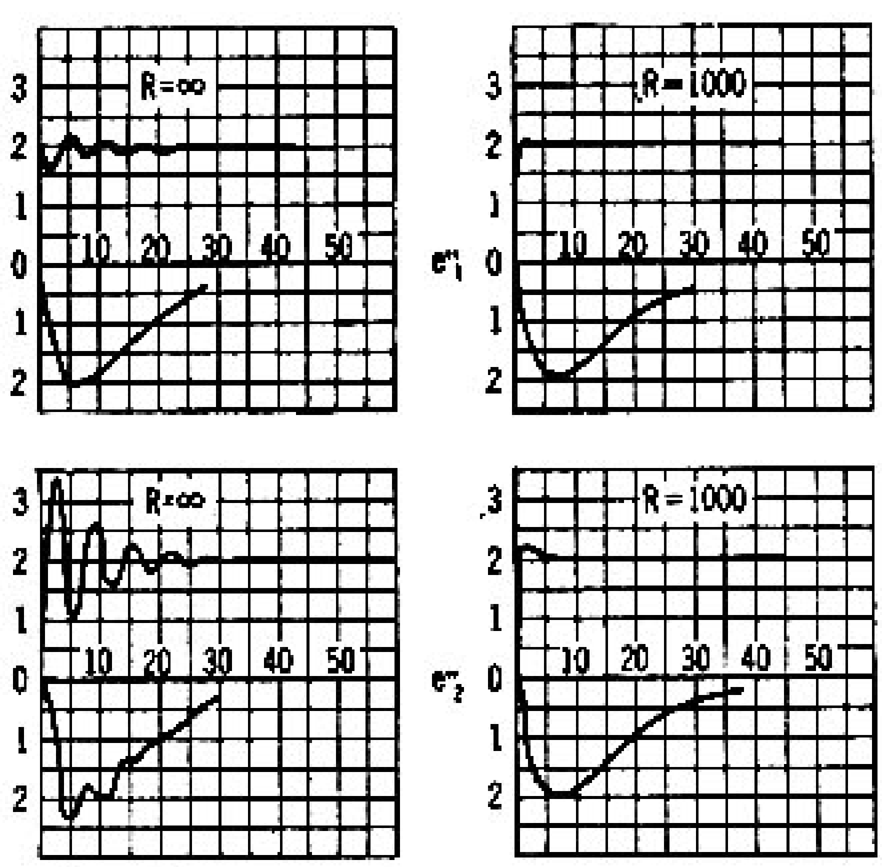


FIG. 42.—$r = 100$, $L = 0.002$, $Z_1 = 400$, $Z_2 = \infty$, $C = 0.0005 \times 10^{-6}$

These curves show that, when the reactor is not equipped with a shunt resistor and no other feeders are connected to the bus ($Z_2 = \infty$), there is danger of very high oscillatory voltages appearing on the bus, approaching a magnitude equal to 400 per cent of the crest of the incident traveling wave. When the incident wave front is lengthened, the magnitude of the oscillatory voltages is greatly reduced. If a suitable resistor (R = 1000 ohms) is shunted across the reactor, the oscillations vanish. In the case illustrated, the stresses on the bus

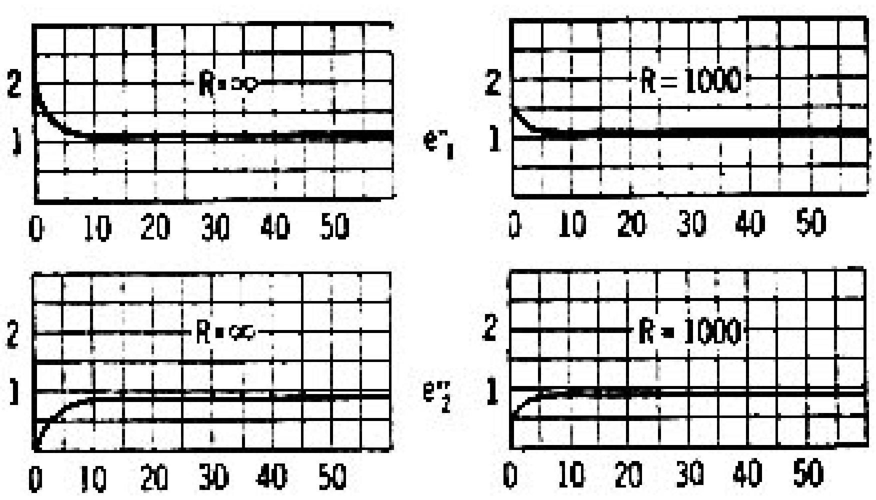


FIG. 43.—r = 100, L = 0.002, Z_1 = 400, Z_2 = 400, $C = 0.0005 \times 10^{-6}$

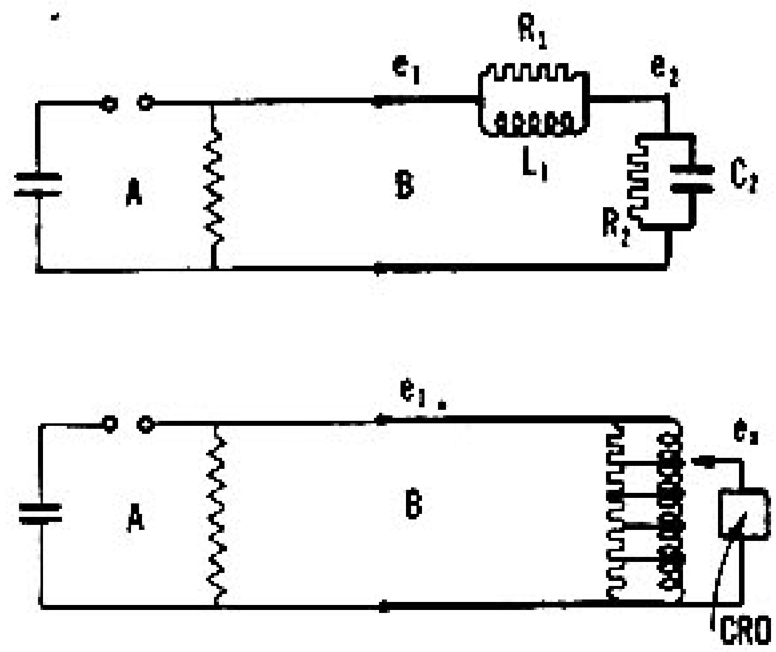


FIG. 44.—Test Circuits
A. Impulse generator
B. Circuit under test

are reduced some 40 per cent by the addition of the shunt resistor. But if one or more additional feeders are connected to the bus, Fig. 43, the surge impedance thereof suffices to damp out the oscillations without the assistance of a shunt resistor. The shunt resistor will, however, reduce the voltage on the line side of the reactor, and may thereby prevent a flashover of the line insulators. Fig. 44 shows the laboratory set-up employed for making impulse studies on reactor circuits, and corresponds to the simplified system of Fig. 39, where the surge impedance Z_2 has been replaced by resistance R_2. Typical cathode-ray oscillograms and calculations are given in Figs. 45, 46, 47, 48, and 49, the latter having been taken with an actual transmission line. These oscillograms illustrate the effect of the shunt resistor in

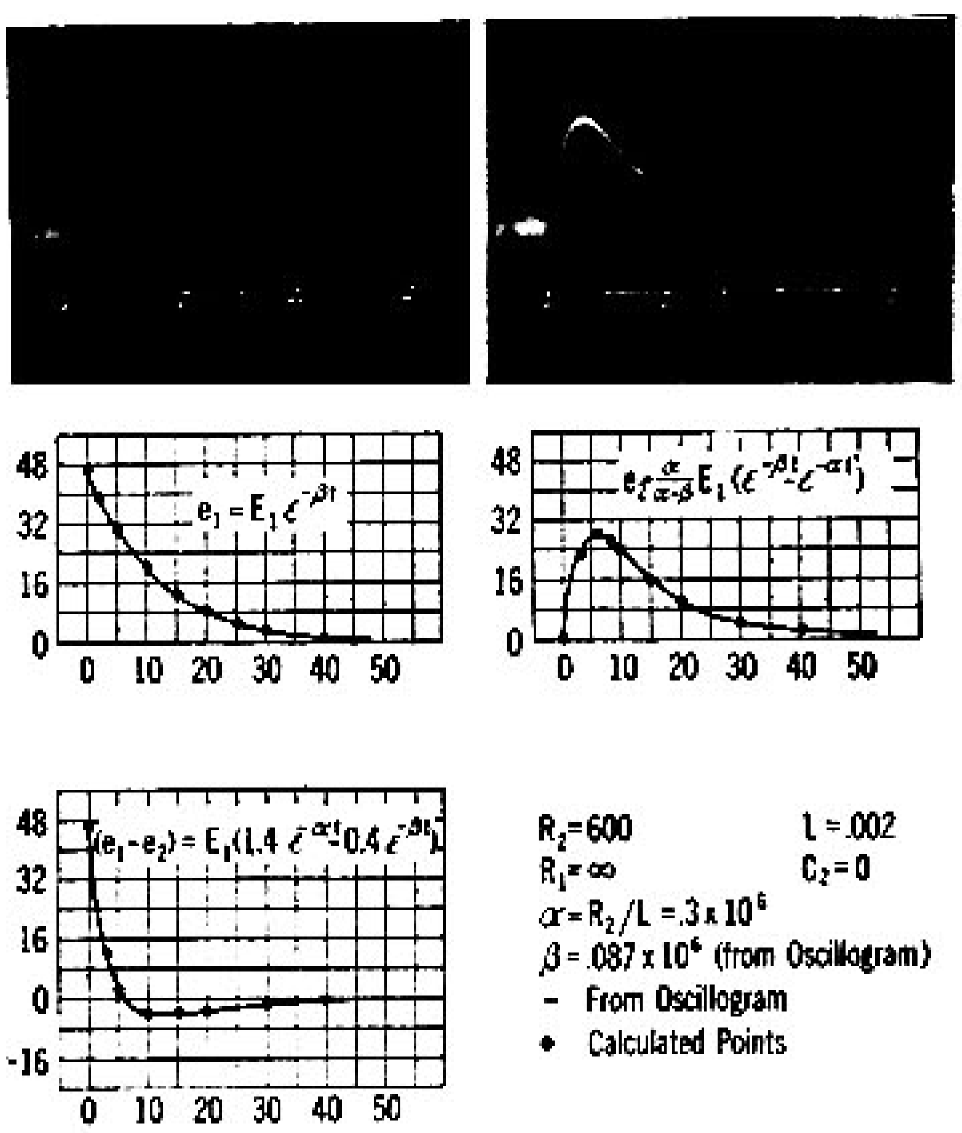


FIG. 45.—Impulse Tests on Reactor—No Shunt Resistor

completely damping out the oscillations. The almost perfect check between the oscillograms and the calculations will be noticed. In Fig. 45 there are no oscillations because $C_2 = 0$ for that case.

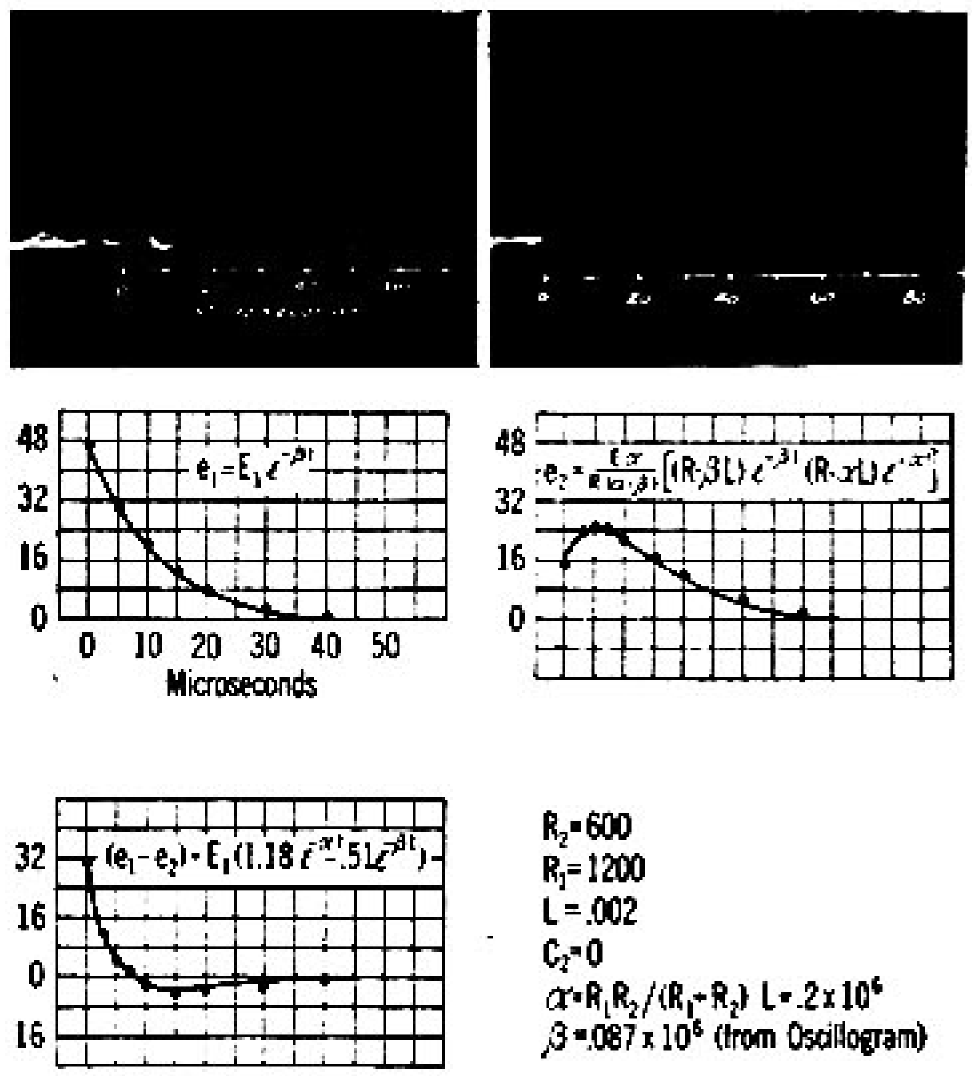


FIG. 46.—Impulse Tests on Reactor—With Shunt Resistor

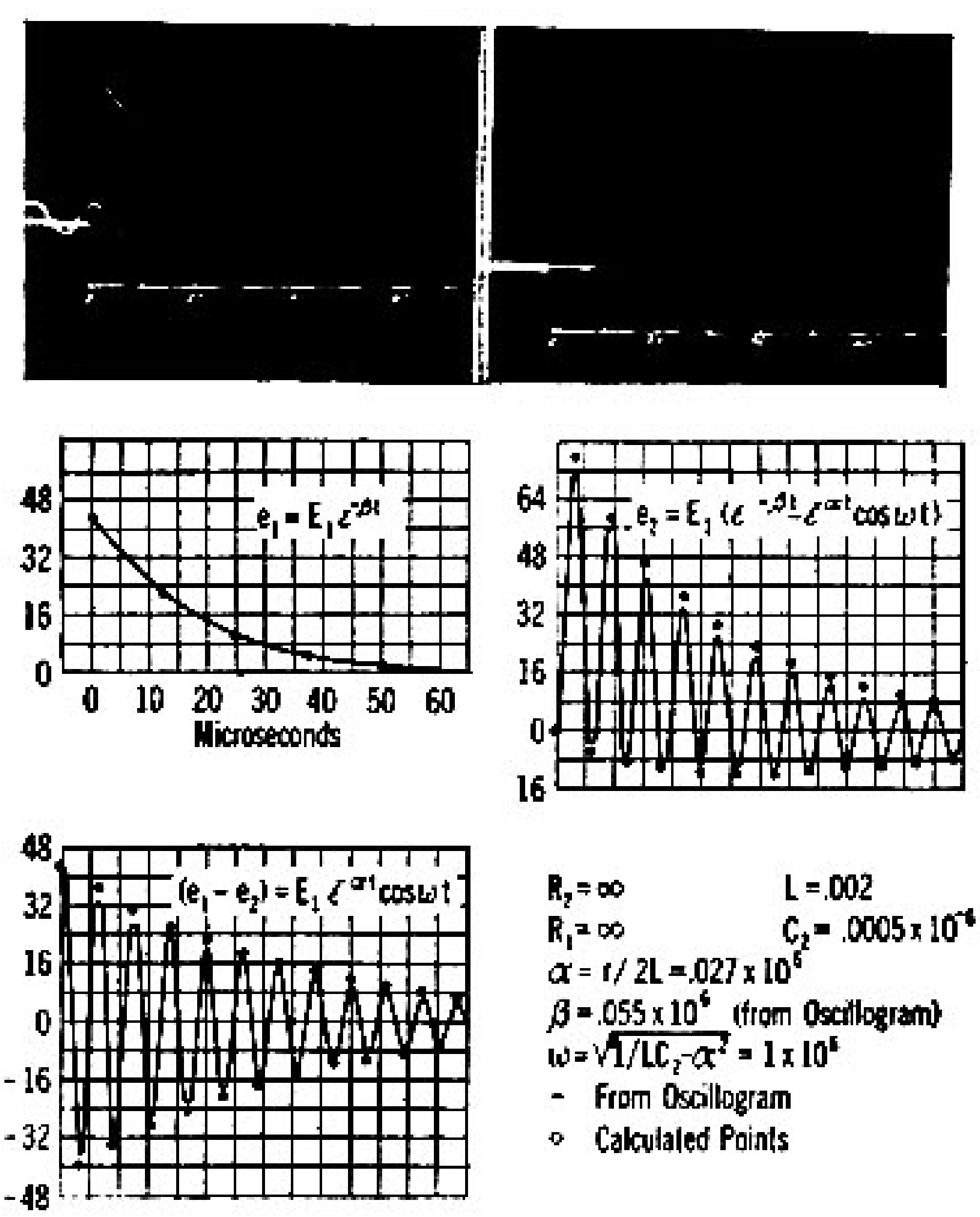


FIG. 47.—Impulse Tests on Reactor—No Shunt Resistor

Figs. 50 and 51 show the internal distribution of voltage along the windings of a reactor with different shunt resistors. When there is no shunt resistor ($R = \infty$), high-frequency internal oscillations are present, caused by the periodic transfer of energy between the inductance and capacitance elements of the winding. In subsequent chapters, the theory of internal oscillations in distributed windings will be discussed in detail. For the present it is convenient merely

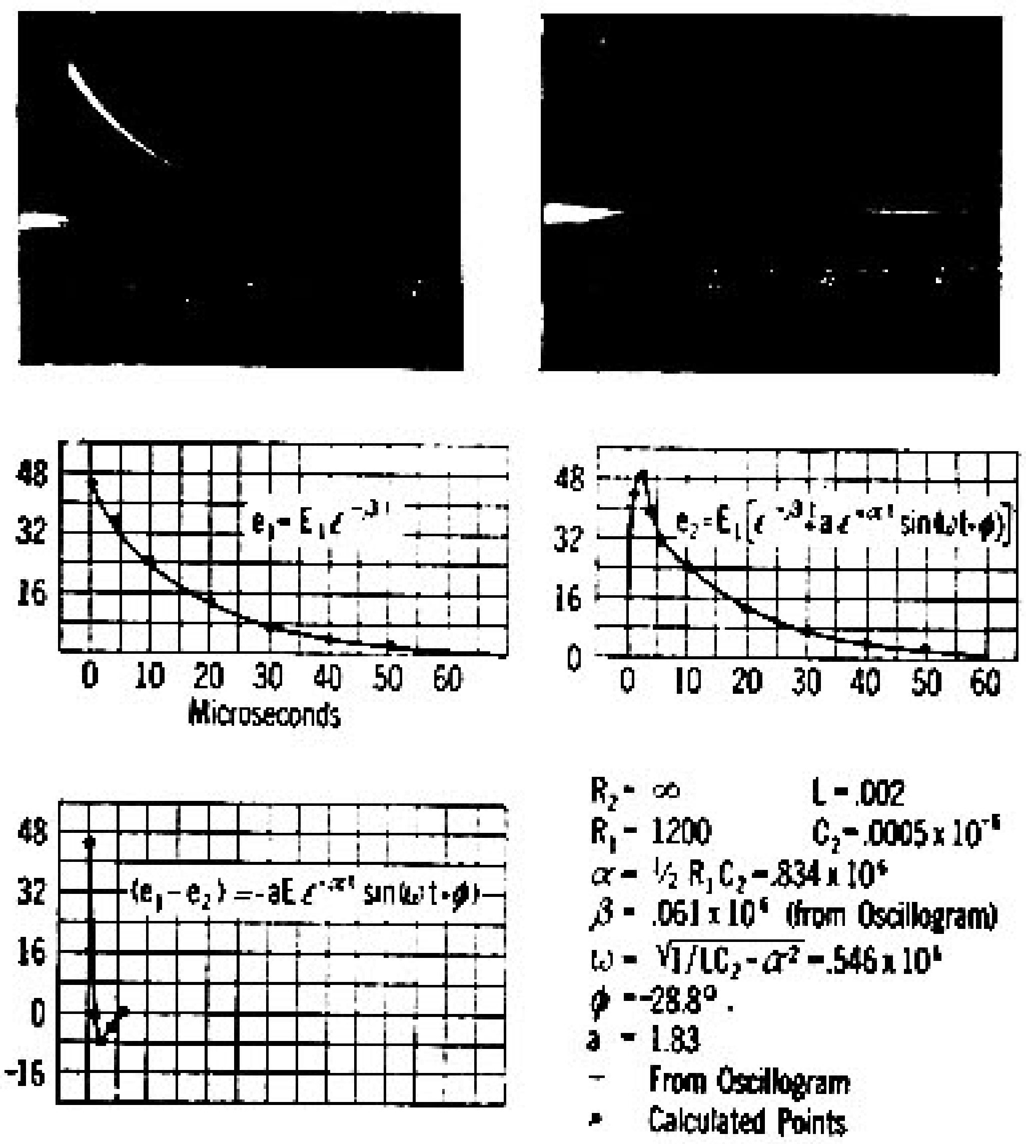


FIG. 48.—Impulse Tests on Reactor—With Shunt Resistor

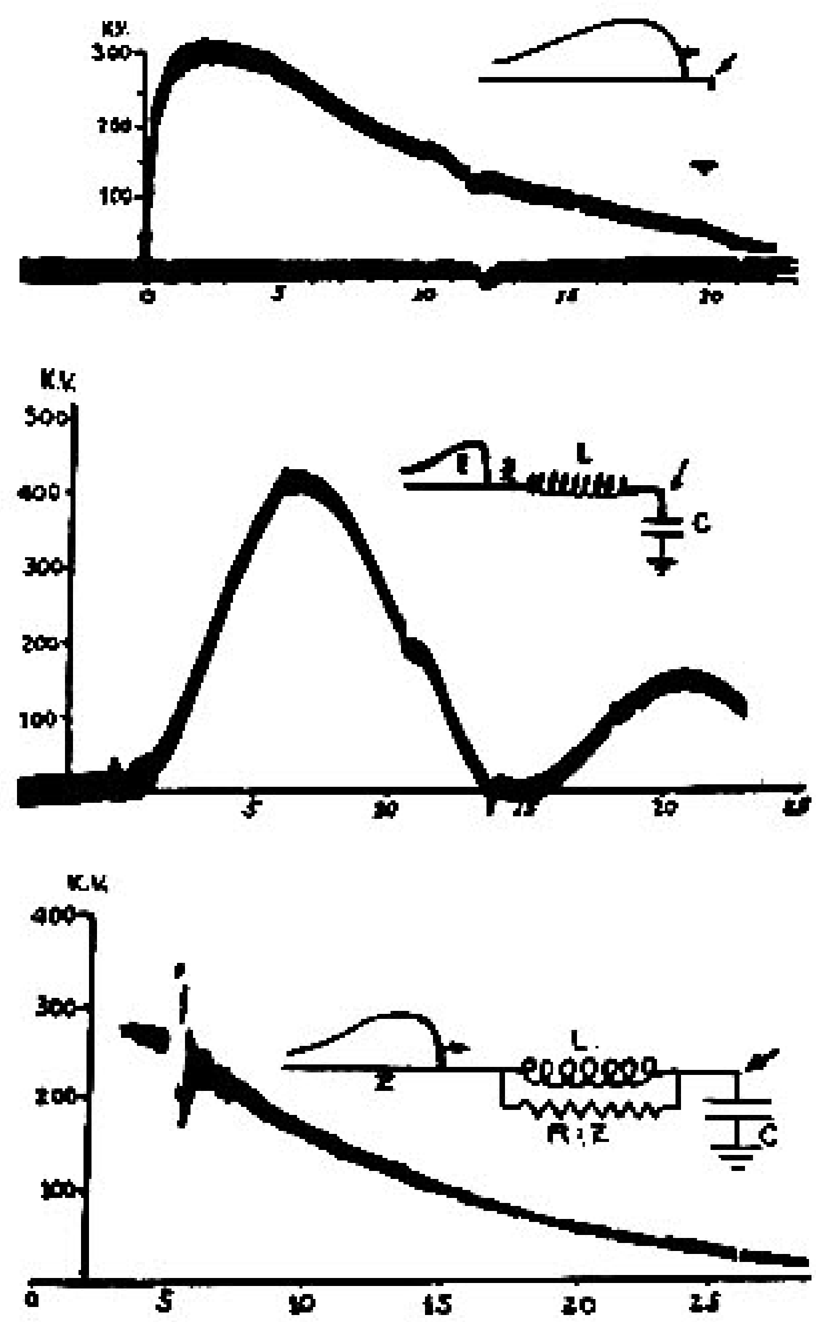


FIG. 49.—Impulse Tests with Transmission Line

Top: Open-circuited line

Middle: Reactor in series with transformer

Bottom: Reactor shunted by a resistor

to point out that such oscillations exist, and that they can be wiped out by a tied-in shunt resistor of sufficiently low resistance.

It is apparent from the above discussion that a resistor in shunt with a reactor has three beneficial results:

1. The reactor can not enter into oscillation with the capacitance of a bus or connected apparatus, thereby mitigating to a large extent the possibility of resonate or cumulative oscillations in the windings of transformers and rotating machines, as

well as greatly reducing the voltage which can appear on the bus.

2. The voltage on the line side of the reactor is reduced at the instant of impact, and the possibility of a line insulator flashover on account of positive reflection is less.
3. Internal oscillations between the inductance and capacitance elements of the reactor winding are destroyed, and the turn-

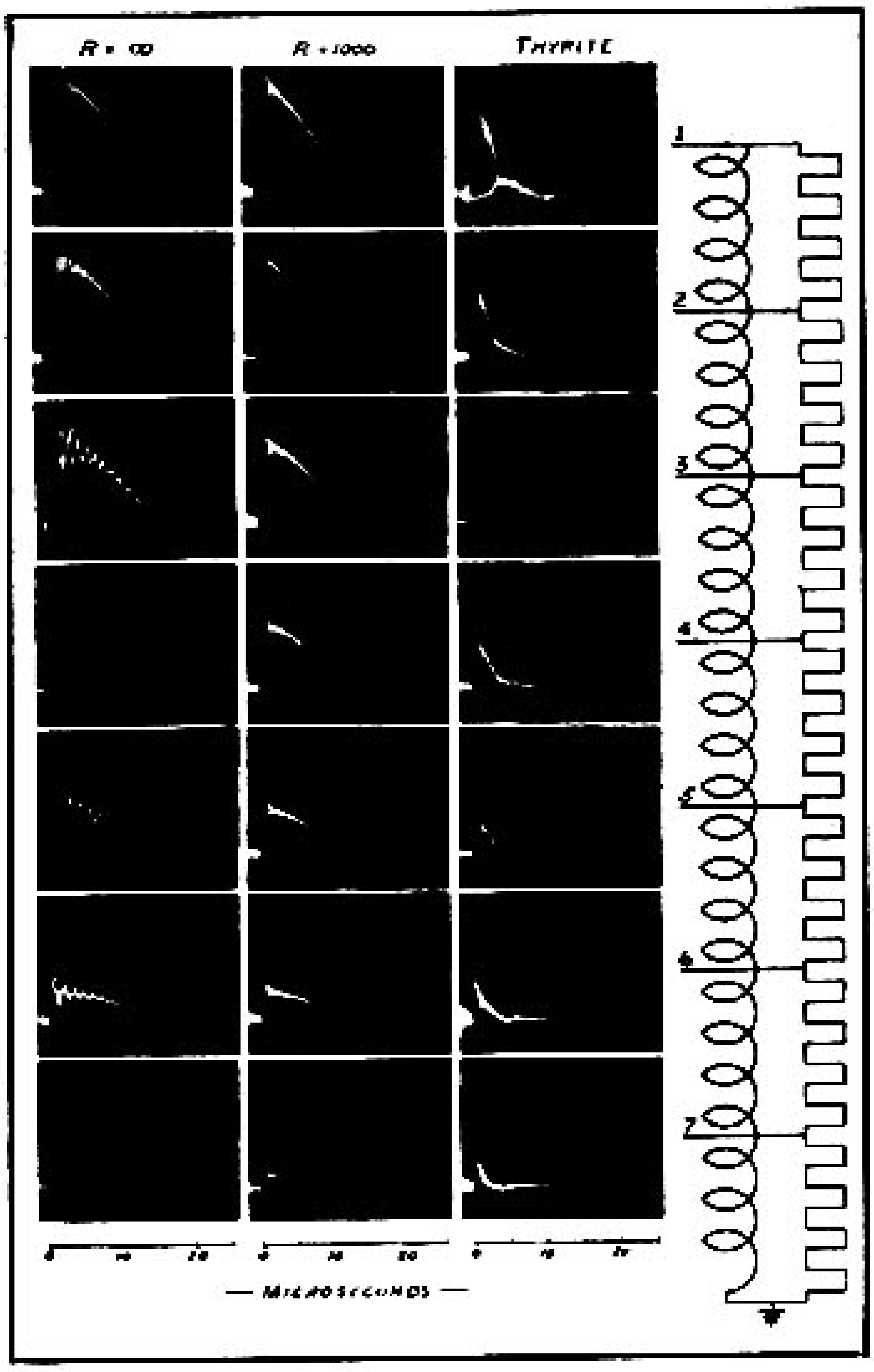


FIG. 50.—Oscillograms Showing Internal Voltage Distribution in Reactors

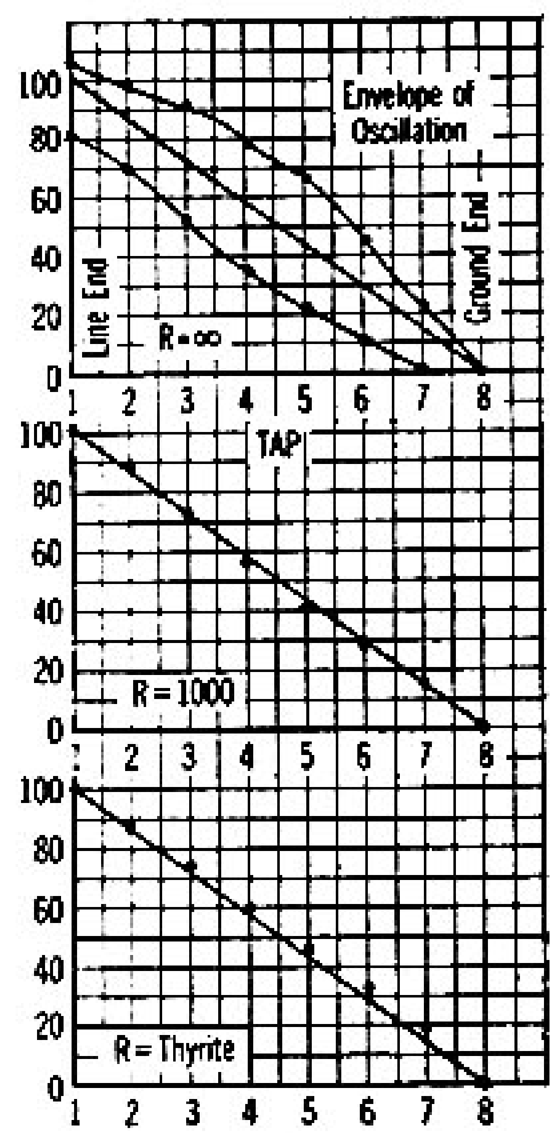


FIG. 51.—Effect of Tied-In Shunt Resistor on Internal Voltage Distribution

to-turn stresses to which the winding is subjected are correspondingly relieved.

Two essential conditions are to be met in the design of shunt resistors for reactors. First, in order to give effective protection against lightning or other high-potential surges, the resistance should not be more than 400 ohms for overhead lines, or 50 ohms for underground cables. Second, the resistor must not overheat when the reactor it is used with is undergoing short circuit. (The purpose of the reactor is to limit the short-circuit current.)

To meet the latter condition, a high resistance is necessary. The rate at which energy is absorbed by the resistor is $P = E^2/R$, where E is the voltage across the resistor and R is its resistance. Thus, in a 13,800-volt circuit, the voltage across the reactor during short circuit is $13{,}800/\sqrt{3} = 7980$ volts, and if the reactor is for use with an underground cable, the resistor (in order to have maximum effectiveness in reducing transient voltage) should have a resistance of the order of 50 ohms. The energy absorbed would then be at the rate of 1274 kw. A resistor capable of absorbing so much energy without overheating would be prohibitive in cost. For these reasons, a material having lower resistance at high voltages than at low voltages is essential. For instance, if the resistor is 1000 ohms with 7980 volts rms. across it and 50 ohms for 35-kw. crest (which is approaching the danger zone for a 13.8 kv. circuit), it has the proper characteristic for the above case. These conditions are admirably fulfilled by Thyrite. Thus, for the standard disks,

$$R = \frac{nC}{I^a} = E\left(\frac{nC}{E}\right)^{\frac{1}{1-a}} = E\left(\frac{580\,n}{E}\right)^{3.57} \tag{119}$$

At $E = 35{,}000$ and $n = 10$ this gives $R = 56.5$ ohms; for $E = \sqrt{2} \times 7980 = 11{,}300$ it gives $R = 1050$ ohms.

SUMMARY OF CHAPTER V

The protection afforded electrical apparatus depends upon the characteristics of the transmission circuit, of the terminal equipment, and of the protective devices. These protective devices aim at control of the *crest*, *front*, and *length* of traveling waves. Schemes for the protection of electrical equipment from high-voltage surges contemplate:

1. Control of the surge at its source, through the use of ground wires, low tower footing resistances, sufficient line insulation, insulator arresters, expulsion fuses, etc. (See Chapter X.)
2. Control of the surge near its point of impact on station apparatus, through the use of insulation coordination, gaps, lightning arresters, by-passes, capacitors, choke coils, Petersen coils (see Chapter XI), surge absorbers, etc.
3. Control of the surge inside the apparatus, through the use of electrostatic shields, tied-in shunt resistors, adequate insulation rationally disposed, proper neutral impedances, etc. (See Chapter XVI.)

The present chapter dealt briefly with a few of the schemes under (2) above, that is, the control of the surge at its point of impact with station apparatus. Effective coordination of system insulation is based upon taking advantage of the impulse breakdown, or sparkover characteristics of gaps, lightning arresters, insulators, bushings, etc., in such a way that the several voltage levels established thereby successively protect the most vital and least accessible parts of the system, just as in a defensive battle position the ground is organized with an outpost area, a main line of resistance, a support line, a battalion reserve line, and a regimental reserve line.

The basic voltage level is established by the coordinating gap, and its sparkover characteristic is therefore the reference level on which the designer bases the rest of the system insulation.

The impulse sparkover characteristic of gaps, insulators, etc., has the general shape:

$$e = e_0\left(1 + \frac{a}{\sqrt{t}}\right)$$

in which e_0 is the 60-cycle breakdown and a is an empirical constant depending upon the type of gap and the applied wave shape.

But a gap by itself can not interrupt the normal power frequency follow-current, and therefore a lightning arrester should be provided whose function is to limit the impulse voltage to values such that the gap does not flash over. Most commercial lightning arresters consist of a gap (or gaps) in series with a material having either a decided negative volt-ampere characteristic, or definite valve action, and this characteristic usually precludes an explicit mathematical equation, so that the solution must be obtained by approximate graphical or step-by-step methods from the equation:

$$2\,e = e_R + z\,i$$

in which the voltage across the arrester (e_R) is itself a function of the arrester current i. The turn-to-turn and coil-to-coil stresses to which the windings of transformers and rotating machines are subjected by traveling waves can be greatly relieved by increasing the wave fronts. On low-voltage circuits, say up to 25-kv., this can be economically and effectively done by capacitors in shunt. Thus a synchronous generator may be protected by a lightning arrester to limit the crest of the applied wave, a capacitor to increase the wave front, and a neutral impedance to prevent reflections.

Choke coils of the conventional size are ineffective in reducing the transmitted wave crest by more than a few per cent, and are of little use beyond increasing an abrupt wave front one or two microseconds. However, they may become a source of resonate oscillations by entering into oscillation with the capacitance of connected apparatus, unless bridged by a resistor.

Current limiting reactors have several hundred times the inductance of the conventional choke coil, and they should, therefore, be shunted by Thyrite in order to prevent oscillations, act as a by-pass to the surge, and improve the internal distribution in the reactor winding. Thyrite shunts have also been used as by-passes on current transformers, for bridging transformer ratio adjusters, and similar applications.

CHAPTER VI

TRAVELING WAVES ON MULTI-CONDUCTOR SYSTEMS *

The conventional treatment of transmission-line transients given in the previous chapters is based on consideration of a single wire and its return, and ignores the presence of other conductors. However, there are many cases in the study of traveling waves where the effect of the other conductors can not be neglected. Sometimes their influence is so vital as completely to change the characteristics of the phenomenon, and entirely erroneous results are obtained if they are not considered. Problems of this type are of special engineering interest in connection with the design of ground wires and other protective schemes, and in general, in the study of mutual effects due to traveling waves.

Maxwell's Electrostatic Coefficients.—Consider a system of n conductors of any shape and arrangement, fixed in position and uncharged. If a unit charge be placed on conductor r suppose that the potentials acquired by the conductors are $(p_{r1}, p_{r2}, \ldots, p_{rr} \ldots, p_{rn})$. Had a charge Q_r been placed on conductor r, instead of unit charge, the potentials would have been Q_r times as large, or $(Q_r p_{r1}, \ldots, Q_r p_{rn})$. In this notation the first subscript denotes the conductor on which the charge was placed (in this case r), and the second subscript the conductor taking that potential. It follows by the principle of superposition that the effect of simultaneous charges $(Q_1, Q_2, \ldots, Q_n)$ on all the n conductors is to give rise to the system of potentials

$$\left.\begin{aligned} e_1 &= p_{11} Q_1 + p_{21} Q_2 + \ldots + p_{n1} Q_n \\ e_2 &= p_{12} Q_1 + p_{22} Q_2 + \ldots + p_{n2} Q_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ e_n &= p_{1n} Q_1 + p_{2n} Q_2 + \ldots + p_{nn} Q_n \end{aligned}\right\} \qquad (120)$$

* "Induktionswirkungen von Wanderwellen in Nachbarleitungen," by K. W. Wagner, E.T.Z., 1914. "Critique of Ground Wire Theory," by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 49. "Traveling Waves on Transmission Systems," by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 50.

The above equations give the potentials in terms of the charges and the coefficients p. These linear coefficients of proportionality depend only on the geometrical properties of the conductors, as their size, shape, and position. In a few simple cases, such as parallel cylinders, they can be calculated, but in most cases they must be determined experimentally.

Solving Equation (120) for the charges Q_r, there is

$$\left.\begin{aligned} Q_1 &= K_{11}e_1 + K_{12}e_2 + \ldots, + K_{1n}e_n \\ Q_2 &= K_{21}e_1 + K_{22}e_2 + \ldots, + K_{2n}e_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Q_n &= K_{n1}e_1 + K_{n2}e_2 + \ldots, + K_{nn}e_n \end{aligned}\right\} \tag{121}$$

where

$$K_{rs} = (-1)^{r+s}\frac{(\text{minor of } D \text{ for which the cofactor is } p_{rs})}{D} \tag{122}$$

and

$$D = \begin{vmatrix} p_{11} & p_{21} & \ldots & p_{n1} \\ p_{12} & p_{22} & \ldots & p_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ p_{1n} & p_{2n} & \ldots & p_{nn} \end{vmatrix} \tag{123}$$

To show that all the potential coefficients p are positive, suppose that $Q_1 = +1$ and $Q_2 = Q_3 = \ldots = Q_n = 0$. Then e_1 is the greatest potential in the field, and the potentials of the other conductors must lie intermediate between e_1 and zero, hence all positive. If $e_1 = +1$ and $e_2 = e_3 = \ldots = e_n = 0$, then it is evident that $K_{11} = Q_{11}$ is positive and K_{12}, K_{23}, etc., are all negative, since the lines from Q_{11} must terminate either at infinity or on the other conductors. Therefore K_{rs} is negative if $r \neq s$. By means of Green's reciprocal theorem (see "Electricity and Magnetism," by J. H. Jeans, p. 92), it may be shown that $p_{rs} = p_{sr}$ and therefore $K_{rs} = K_{sr}$. Recapitulating,

$$\left.\begin{aligned} &p_{rs} = p_{sr} \text{ and } K_{rs} = K_{sr} \\ &p_{rr},\ p_{rs},\ K_{rr} \text{ are positive} \\ &K_{rs} \text{ is negative if } r \neq s \end{aligned}\right\} \tag{124}$$

Now consider a system of parallel cylindrical conductors which are of sufficient length so that end effects are negligible, and whose spacings are large compared to their radii. If these conductors are over a zero potential plane, as shown in Fig. 52, then the method of images may be applied, and from electrostatics there is

$$p_{rr} = 2 \log \frac{2h}{\rho} \times 9 \times 10^{-11} \text{ daraf per cm} \tag{125}$$

$$p_{rs} = 2 \log \frac{a}{b} \times 9 \times 10^{-11} \text{ daraf per cm.} \tag{126}$$

where

ρ = radius of conductor r.

h = height of conductor r above the zero potential plane.

a = distance between r and the image of s.

b = distance between r and s.

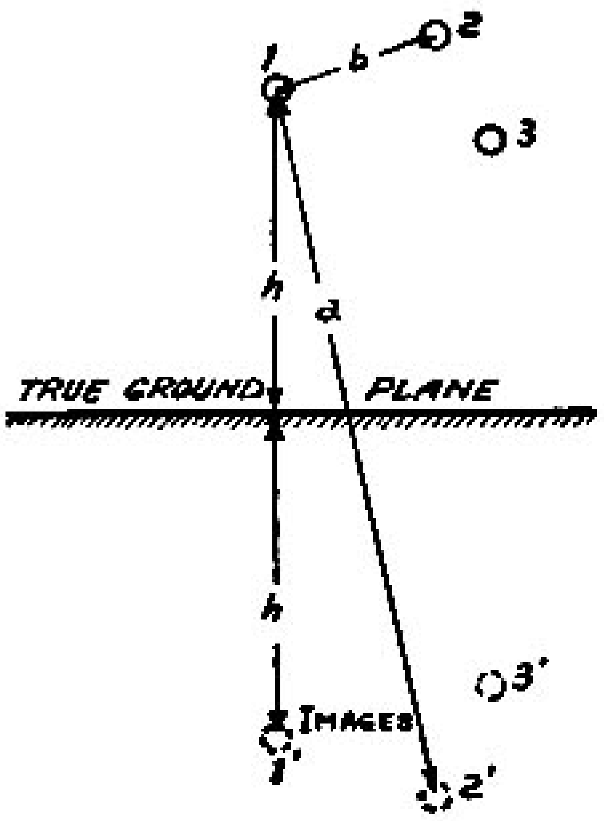


FIG. 52.—System of Conductors and Their Images

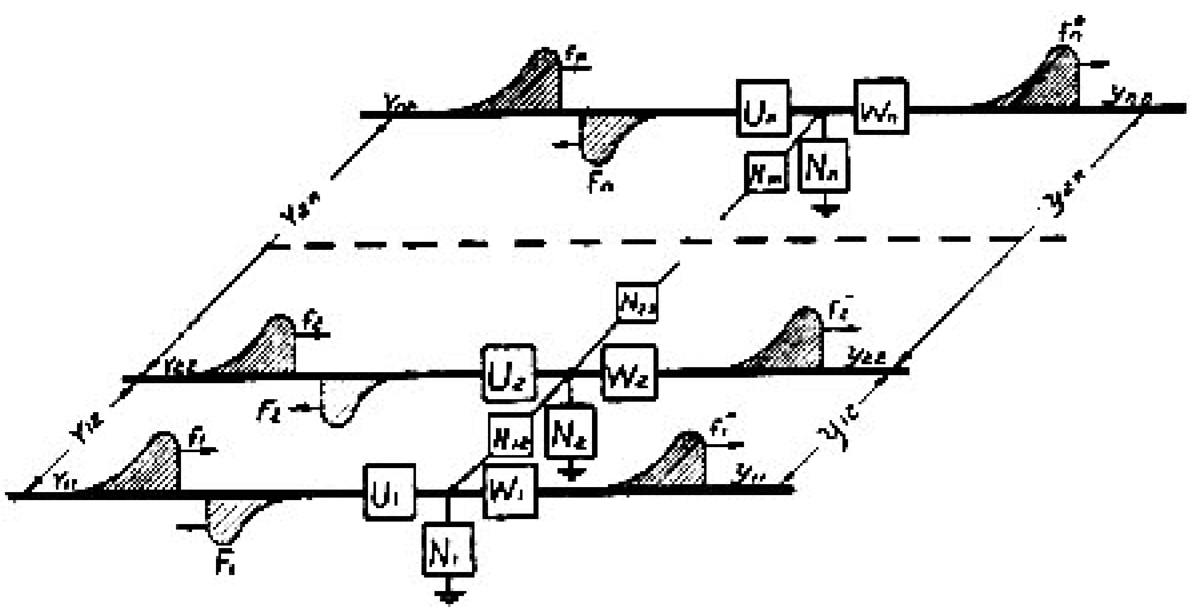


FIG. 53.—General Multi-Conductor System

Maxwell's Electromagnetic Coefficients.—If, in a system of n conductors carrying currents and free from saturation effects, the flux linkage of conductor r due to its own current i_r is $(L_{rr}\, i_r)$, and the flux linkage of conductor s due to the current i_r is $(L_{rs}\, i_r)$, then by the principle of superposition the total flux linkages for all conductors are

$$\left.\begin{aligned} \phi_1 &= L_{11}\, i_1 + L_{12}\, i_2 + \ldots L_{1n}\, i_n \\ \phi_2 &= L_{21}\, i_1 + L_{22}\, i_2 + \ldots L_{2n}\, i_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_n &= L_{n1}\, i_1 + L_{n2}\, i_2 + \ldots L_{nn}\, i_n \end{aligned}\right\} \tag{127}$$

From energy considerations (see "Electricity and Magnetism," by J. H. Jeans, p. 443), it may be shown that

$$L_{rs} = L_{sr} \tag{128}$$

For the system of parallel wires of Fig. 52

$$L_{rr} = \left(\frac{1}{2} + 2 \log \frac{2\,h}{\rho}\right) 10^{-9} \text{ henry per cm.}$$

$$L_{rs} = \left(2 \log \frac{a}{b}\right) 10^{-9} \text{ henry per cm.} \tag{129}$$

The General Differential Equations of Traveling Waves.—Fig. 53 shows a system of n transmission-line conductors, parallel to each other and to the ground plane, and mutually coupled electromagnetically and electrostatically, so that the effects of currents and potentials on any wire are felt on all the other wires. The circuit constants involved are shown in Fig. 54. Associated with each unit length of line and conductors r and s there is

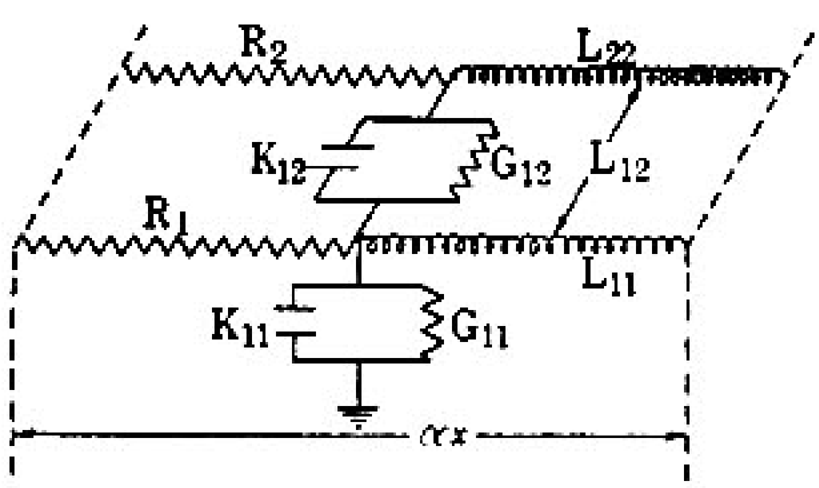


FIG. 54.—Circuit Constants of Mutually Coupled Circuits

L_{rr} = self-inductance coefficient of conductor r.

L_{rs} = mutual-inductance coefficient between r and s.

K_{rr} = self-capacitance coefficient of conductor r.

K_{rs} = mutual-capacitance coefficient between r and s.

R_r = series resistance of conductor r.

g_{rr} = leakage conductance to ground of conductor r.

g_{rs} = leakage conductance between r and s.

It will also be convenient to introduce the notation

$$G_{rr} = (g_{r1} + g_{r2} + g_{r3} + \dots + g_{rn})$$

$$G_{rs} = G_{sr} = -\,g_{rs} = -\,g_{sr}$$

$$Z_{rr} = (R_r + p\,L_{rr})$$

$$Z_{rs} = p\,L_{rs}$$

$$Y_{rr} = (G_{rr} + p\,K_{rr})$$
$$Y_{rs} = (G_{rs} + p\,K_{rs})$$
$$p \;= \partial \; \partial t$$

The charges per unit length on the conductors in terms of Maxwell's electrostatic coefficients and the potentials $e_1, e_2, \ldots, e_n$ are given by Equation (121).

The magnetic fluxes linking the conductors per unit length in terms of the inductance coefficients and the currents $i_1, i_2, \ldots, i_n$ are given by Equation (127).

The leakage currents flowing to ground and to the other conductors are

$$\left.\begin{aligned} i_1' &= g_{11}\,e_1 + g_{12}\,(e_1 - e_2) + \ldots + g_{1n}\,(e_1 - e_n) \\ &= G_{11}\,e_1 + G_{12}\,e_2 + \ldots + G_{1n}\,e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ i_n' &= g_{n1}\,(e_n - e_1) + g_{n2}\,(e_n - e_2) + \ldots + g_{nn}\,e_n \\ &= G_{n1}\,e_1 + G_{n2}\,e_2 + \ldots + G_{nn}\,e_n \end{aligned}\right\} \tag{130}$$

The differential equations of the first conductor are

$$-\frac{\partial e_1}{\partial x} = \frac{\partial \phi_1}{\partial t} + R_1\,i_1 = Z_{11}\,i_1 + Z_{12}\,i_2 + \ldots + Z_{1n}\,i_n \tag{131}$$

$$-\frac{\partial i_1}{\partial x} = \frac{\partial Q_1}{\partial t} + i_1' = Y_{11}\,e_1 + Y_{12}\,e_2 + \ldots + Y_{1n}\,e_n \tag{132}$$

Differentiating Equation (131) with respect to x and substituting the equations of type (132) there is

$$\begin{aligned} \frac{\partial^2 e_1}{\partial x^2} &= Z_{11}\,(Y_{11}\,e_1 + Y_{12}\,e_2 + \ldots + Y_{1n}\,e_n) \\ &\quad + Z_{12}\,(Y_{21}\,e_1 + Y_{22}\,e_2 + \ldots + Y_{2n}\,e_n) \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + Z_{1n}\,(Y_{n1}\,e_1 + Y_{n2}\,e_2 + \ldots + Y_{nn}\,e_n) \\ &= (Z_{11}\,Y_{11} + Z_{12}\,Y_{21} + \ldots + Z_{1n}\,Y_{n1})\,e_1 \\ &\quad + (Z_{11}\,Y_{12} + Z_{12}\,Y_{22} + \ldots + Z_{1n}\,Y_{n2})\,e_2 \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + (Z_{11}\,Y_{n1} + Z_{12}\,Y_{2n} + \ldots + Z_{1n}\,Y_{nn})\,e_n \\ &= J_{11}\,e_1 + J_{12}\,e_2 + \ldots + J_{1n}\,e_n \end{aligned} \tag{133}$$

where

$$J_{rs} = Z_{r1} Y_{1s} + Z_{r2} Y_{2s} + Z_{r3} Y_{3s} + \ldots + Z_{rn} Y_{ns}$$

$$= Z_{1r} Y_{1s} + Z_{2r} Y_{2s} + Z_{3r} Y_{3s} + \ldots + Z_{nr} Y_{ns} \tag{134}$$

Let

$$A_{rr} = \left(J_{rr} - \frac{\partial^2}{\partial x^2}\right) \tag{135}$$

Then the complete set of differential equations for the n conductors may be written in the symbolic form

$$\left.\begin{array}{l} 0 = A_{11} e_1 + J_{12} e_2 + \ldots + J_{1n} e_n \\ 0 = J_{21} e_1 + A_{22} e_2 + \ldots + J_{2n} e_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = J_{n1} e_1 + J_{n2} e_2 + \ldots + A_{nn} e_n \end{array}\right\} \tag{136}$$

where the J's are operators in the time derivative $p = \partial/\partial t$, and the A's are operators in both the time and space derivatives. Solving these n simultaneous equations for any e, there results a determinate of which the numerator is zero on account of having a column of zeros. In order, therefore, that a solution other than zero can exist, it is necessary that the denominator also be zero (on the assumption that the indeterminate so formed will evaluate to a finite value). Therefore, there must be

$$\begin{vmatrix} A_{11} & J_{12} & \ldots & J_{1n} \\ J_{21} & A_{22} & \ldots & J_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ J_{n1} & J_{n2} & \ldots & A_{nn} \end{vmatrix} e = 0 \tag{137}$$

Now, dropping subscripts

$$Z\,Y = (R + pL)\,(G + pK) = L\,K\,p^2 + (L\,G + R\,K)\,p + R\,G \tag{138}$$

so that, ultimately, the expansion of (137) will lead to a polynomial of degree n in $(\partial^2/\partial x^2)$ and degree n in $p^2 = \partial^2/\partial t^2$. The solution of this partial differential equation is the most general solution for a system of parallel conductors.

There are three conditions under which Equation (137) may be considerably simplified before attempting a solution. These are: I, The No-Loss Line; II, The Completely Transposed Line; and III, The Alternating-Current Solution.

Case I. The No-Loss Line.—If there are no losses, then in (138) $R = G = 0$ and $ZY = p^2 LK$, whereupon

$$J_{rs} = p^2 (L_{1r} K_{1s} + L_{2r} K_{2s} + \dots + L_{nr} K_{ns}) = p^2 I_{rs} \quad (139)$$

$$A_{rr} = \left(p^2 I_{rr} - \frac{\partial^2}{\partial x^2}\right) \quad (140)$$

Now if e is tentatively *assumed* to be a traveling wave

$$e = f(x + vt) \quad (141)$$

Then since

$$\left.\begin{aligned} J_{rs}\, e &= v^2 I_{rs} f''(x + vt) \\ A_{rr}\, e &= (v^2 I_{rr} - 1) f''(x + vt) = v^2 B_{rr} f''(x + vt) \end{aligned}\right\} \quad (142)$$

it follows that (137) becomes

$$\begin{vmatrix} v^2 B_{11} & v^2 I_{12} \dots & v^2 I_{1n} \\ v^2 I_{21} & v^2 B_{22} \dots & v^2 I_{2n} \\ \cdot & \cdot & \cdot \\ v^2 I_{n1} & v^2 I_{n2} \dots & v^2 B_{nn} \end{vmatrix} f^{(2n)}(x + vt) = 0 \quad (143)$$

Dividing (143) by $v^{2n} f^{(2n)}(x + vt)$ there results

$$\begin{vmatrix} B_{11} & I_{12} \dots & I_{1n} \\ I_{21} & B_{22} \dots & I_{2n} \\ \cdot & \cdot & \cdot \\ I_{n1} & I_{n2} \dots & B_{nn} \end{vmatrix} = 0 \quad (144)$$

which is the equation to be satisfied by the velocity v, in which $B_{rr} = (I_{rr} - v^{-2})$. The determinate (144) is of order n, and therefore, its expansion will yield a polynomial of degree n in v^{-2}. It follows, therefore, that there are n independent values of v^2 which satisfy (144), and consequently on an n-conductor system there are $2\,n$ values for the velocity of propagation (n positive and n negative) which satisfy the conditions for wave motion, and these values are given by the roots of (144). These considerations show that, in general, there can exist simultaneously on each conductor of an n-conductor system n pairs of waves of different velocities of propagation ($v_1, v_2, \dots, v_n$) and each pair consists of a forward and backward wave. Thus

$$\begin{aligned} e_1 = [f_{11}(x - v_1 t) + F_{11}(x + v_1 t)] + \dots \\ + [f_{1n}(x - v_n t) + F_{1n}(x + v_n t)] \end{aligned} \quad (145)$$

The current waves follow from (132) upon integrating partially with respect to x, and remembering that $Y = pK$ in the no-loss line,

$$
\begin{aligned}
i_1 &= -\frac{\partial}{\partial t}\int (K_{11}\, e_1 + K_{12}\, e_2 + \ldots + K_{1n}\, e_n)\, dx \\
&= K_{11} \sum v_r (f_{1r} - F_{1r}) + K_{12} \sum v_r (f_{2r} - F_{2r}) + \ldots \\
&\quad + K_{1n} \sum v_r (f_{nr} - F_{nr}) \\
&= \sum v_r [K_{11} (f_{1r} - F_{1r}) + K_{12} (f_{2r} - F_{2r}) + \ldots \\
&\quad + K_{1n} (f_{nr} - F_{nr})]
\end{aligned} \tag{146}
$$

where the summations include all the waves in the expressions such as (145) for the potentials.

For traveling waves due to lightning, the transient skin effect is so high that the current is confined to a thin skin at the periphery of the conductor. Consequently there is no internal magnetic field, and the factor 1/2 in (129) vanishes (it is due to the internal interlinkages on the assumption of uniform current distribution throughout the cross-section of the conductor). Then Equations (129) become

$$L_{rr} = 2 \log \frac{2\,h}{\rho} \times 10^{-9} = \frac{p_{rr}}{c^2} \text{ henrys per cm.} \tag{147}$$

$$L_{rs} = 2 \log \frac{a}{b} \quad \times 10^{-9} = \frac{p_{rs}}{c^2} \text{ henrys per cm.} \tag{148}$$

$$c = (3 \times 10^{10}) \text{ cm. per sec.} = \text{velocity of light} \tag{149}$$

substituting these values in (139), there is

$$I_{rs} = c^{-2} (p_{1r}\, K_{1s} + p_{2r}\, K_{2s} + \ldots + p_{nr}\, K_{ns}) \tag{150}$$

Referring back now to (122), and remembering that $D\,K_{rs}$ is the minor, of which the cofactor is p_{rs} in the expansion of D, it is evident that (150) is that expansion if the elements of the r and s columns are identical. But in such a case a determinate vanishes. Therefore

$$I_{rs} = \begin{cases} 0 \text{ if } r \neq s \\ c^{-2} \text{ if } r = s \end{cases} \tag{151}$$

Under these conditions

$$B_{rr} = (c^{-2} - v^{-2}) = B \tag{152}$$

and the determinate (144) degenerates to

$$\begin{vmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \cdot & & \cdot & \cdot \\ 0 & & \dots & B \end{vmatrix} = B^n = (c^{-2} - v^{-2})^n = 0 \tag{153}$$

Therefore $v = \pm c$, and all waves have the same velocity—that of light.

Hereby Equations (146) reduce to

$$\left.\begin{aligned} i_1 &= Y_{11}(f_1 - F_1) + Y_{12}(f_2 - F_2) + \dots + Y_{1n}(f_n - F_n) \\ i_2 &= Y_{21}(f_1 - F_1) + Y_{22}(f_2 - F_2) + \dots + Y_{2n}(f_n - F_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ i_n &= Y_{n1}(f_1 - F_1) + Y_{n2}(f_2 - F_2) + \dots + Y_{nn}(f_n - F_n) \end{aligned}\right\} \tag{154}$$

where

$$\left.\begin{aligned} Y_{rr} &= c\,K_{rr} = \text{self surge admittance} \\ Y_{rs} &= c\,K_{rs} = \text{mutual surge admittance} \end{aligned}\right\} \tag{155}$$

Inverting the order of solution which led to Equations (154) there are

$$\left.\begin{aligned} e_1 &= Z_{11}(g_1 - G_1) + Z_{21}(g_2 - G_2) + \dots + Z_{n1}(g_n - G_n) \\ e_2 &= Z_{12}(g_1 - G_1) + Z_{22}(g_2 - G_2) + \dots + Z_{n2}(g_n - G_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ e_n &= Z_{1n}(g_1 - G_1) + Z_{2n}(g_2 - G_2) + \dots + Z_{nn}(g_n - G_n) \end{aligned}\right\} \tag{156}$$

where

$$\left.\begin{aligned} Z_{rr} &= \frac{p_{rr}}{c} = 60 \log\left(\frac{2\,h}{\rho}\right) = \text{self surge impedance} \\ Z_{rs} &= \frac{p_{rs}}{c} = 60 \log\left(\frac{a}{b}\right) = \text{mutual surge impedance} \end{aligned}\right\} \tag{157}$$

$$\left.\begin{aligned} g &= g\,(x - vt) = \text{forward current wave} \\ G &= G\,(x + vt) = \text{reverse current wave} \end{aligned}\right\} \tag{158}$$

If

$$D = \begin{vmatrix} Z_{11} & Z_{21} & \dots & Z_{n1} \\ \cdot & \cdot & \cdot & \cdot \\ Z_{1n} & Z_{2n} & \dots & Z_{nn} \end{vmatrix} \tag{159}$$

$$D' = \begin{vmatrix} Y_{11} & Y_{21} & \dots & Y_{n1} \\ \cdot & \cdot & \cdot & \cdot \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix} \tag{160}$$

then the Z's and Y's are related to each other as

$$Y_{rs} = \frac{(-1)^{(s+r)} \text{ (minor of } D \text{ for which the cofactor is } Z_{rs})}{D} \tag{161}$$

$$Z_{rs} = \frac{(-1)^{(s+r)} \text{ (minor of } D' \text{ for which the cofactor is } Y_{rs})}{D'} \tag{162}$$

Considering waves going only in one direction

$$\left.\begin{aligned} \pm e_1 &= Z_{11} i_1 + \dots + Z_{n1} i_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \pm e_n &= Z_{1n} i_1 + \dots + Z_{nn} i_n \end{aligned}\right\} \tag{163}$$

$$\left.\begin{aligned} \pm i_1 &= Y_{11} e_1 + \dots + Y_{n1} e_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \pm i_n &= Y_{1n} e_1 + \dots + Y_{nn} e_n \end{aligned}\right\} \tag{164}$$

where the plus sign is used for waves traveling in the forward direction, and the minus sign is used for waves traveling in the reverse direction.

In practical cases it often happens that a certain group of conductors is constrained to carry equipotential waves, in which case it is convenient to replace the effects of this group by that of a single equivalent conductor. The properties of such an equivalent conductor are defined as follows. Let there be n conductors carrying equipotential waves e_0 and currents $(i_1, i_2, \dots, i_n)$. Then the total current is

$$i_0 = (i_1 + i_2 + \dots + i_n) \tag{165}$$

and the self surge impedance of all wires in parallel is defined as

$$Z_{00} = \frac{e_0}{i_0} = \frac{e_0}{i_1 + i_2 + \dots + i_n} \tag{166}$$

Now putting $e_1 = e_2 = \dots = e_n = e_0$ in (163) and solving for the currents

$$i_1 = \frac{A_1}{D} e_0, \quad i_2 = \frac{A_2}{D} e_0, \dots, \quad i_n = \frac{A_n}{D} e_0 \tag{167}$$

where

$$A_r = \begin{array}{c} \begin{array}{ccc} (1) & (r) & (n) \end{array} \\ \begin{vmatrix} Z_{11} \ldots 1 \ldots Z_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Z_{n1} \ldots 1 \ldots Z_{nn} \end{vmatrix} \end{array} \tag{168}$$

that is, A_r is the same as determinate D with the rth column replaced by a column of *ones*. Then

$$Z_{00} = \frac{D}{A_1 + A_2 + \ldots + A_n} \tag{169}$$

The mutual surge impedance between the group of wires and an independent wire k not of the group is defined as

$$Z_{0k} = \frac{e_k}{i_0} = \frac{Z_{1k}\, i_1 + Z_{2k}\, i_2 + \ldots + Z_{nk}\, i_n}{i_0} \tag{170}$$

where e_k is the voltage induced in k by the group of conductors in question. Hence by (167)

$$Z_{0k} = \frac{Z_{1k}\, A_1 + Z_{2k}\, A_2 + \ldots + Z_{nk}\, A_n}{A_1 + A_2 + \ldots + A_n} \tag{171}$$

As an example, for three conductors let

$$Z_{11} = Z_{22} = Z_{33} = 60 \log \left(\frac{2 \times 12 \times 32}{0.23} \right) = 487$$

$$Z_{12} = Z_{23} = 60 \log \left(\frac{65.6}{14} \right) = 93$$

$$Z_{31} = 60 \log \left(\frac{69.9}{28} \right) = 55$$

$$D = \begin{vmatrix} 487 & 93 & 55 \\ 93 & 487 & 93 \\ 55 & 93 & 487 \end{vmatrix} = 106.5 \times 10^6$$

$$A_1 = \begin{vmatrix} 1 & 93 & 55 \\ 1 & 487 & 93 \\ 1 & 93 & 487 \end{vmatrix} = 17.01 \times 10^4$$

$$A_2 = \begin{vmatrix} 487 & 1 & 55 \\ 93 & 1 & 93 \\ 55 & 1 & 487 \end{vmatrix} = 15.26 \times 10^4$$

$$A_3 = \begin{vmatrix} 487 & 93 & 1 \\ 93 & 487 & 1 \\ 55 & 93 & 1 \end{vmatrix} = 17.01 \times 10^4$$

The equivalent surge impedance of all three wires in parallel therefore is

$$Z_{00} = \frac{D}{A_1 + A_2 + A_3} = \frac{106.5}{17.01 + 15.26 + 17.01} = 216$$

If all the self surge impedances are equal to Z, and all the mutual surge impedances are equal to Z', then from (163)

$$Z_{00} = \frac{Z + (n-1) Z'}{n} \tag{172}$$

If the equalities upon which (172) is based do not hold, nevertheless, by using average values, very close results obtain. Referring to the numerical example above

$$Z_{av} = \frac{Z_{11} + Z_{22} + Z_{33}}{3} = 487$$

$$Z'_{av} = \frac{Z_{12} + Z_{23} + Z_{31}}{3} = 80$$

hence by (172)

$$Z_{00} = \frac{487 + 2 \times 80}{3} = 216$$

If the surge admittances are already known, the equivalent conductor may be conveniently defined in terms of them. Referring to (164), suppose that the group of conductors numbered from $(m+1)$ to n inclusive is carrying equipotential waves. Let

$$e_0 = e_{(m+1)} = \ldots = e_n$$

$$i_0 = i_{(m+1)} + i_{(m+2)} + \ldots + i_n$$

$$Y_{r0} = Y_{(m+1)r} + \ldots + Y_{nr}$$

$$Y_{00} = Y_{(m+1)0} + \ldots + Y_{n0}$$

Then combining the equations $i_{(m+1)}$ to i_n there results

$$\left.\begin{aligned} i_1 &= Y_{11}\, e_1 + \ldots + Y_{m1}\, e_n + Y_{10}\, e_0 \\ &\ldots\ldots\ldots\ldots \\ i_m &= Y_{1m}\, e_1 + \ldots + Y_{mm}\, e_n + Y_{m0}\, e_0 \\ i_0 &= Y_{10}\, e_1 + \ldots + Y_{m0}\, e_n + Y_{00}\, e_0 \end{aligned}\right\} \tag{173}$$

Hereby the number of simultaneous equations has been reduced from n to $(m + 1)$.

The total energy residing in the electrostatic field is

$$\begin{aligned} W_e &= \frac{1}{2}\int (Q_1 e_1 + Q_2 e_2 + \ldots + Q_n e_n)\, dx \\ &= \frac{1}{2}\int (K_{11} e_1^2 + K_{12} e_2 e_1 + \ldots + K_{1n} e_n e_1)\, dx \\ &\ldots\ldots\ldots\ldots \\ &+ \frac{1}{2}\int (K_{n1} e_1 e_n + K_{n2} e_2 e_n + \ldots + K_{nn} e_n^2)\, dx \end{aligned} \tag{174}$$

where the integration is to extend over the lengths of the waves.

The electromagnetic energy (Jeans, p. 443) is

$$\begin{aligned} W_i &= \frac{1}{2}\int (\phi_1\, i_1 + \phi_2\, i_2 + \ldots + \phi_n\, i_n)\, dx \\ &= \frac{1}{2}\int (L_{11}\, i_1^2 + L_{12}\, i_2\, i_1 + \ldots + L_{1n}\, i_n\, i_1)\, dx \\ &\ldots\ldots\ldots\ldots \\ &+ \frac{1}{2}\int (L_{n1}\, i_1\, i_n + L_{n2}\, i_2\, i_n + \ldots + L_{nn}\, i_n^2)\, dx \end{aligned} \tag{175}$$

The total energy of the waves is (considering only the waves moving in one direction),

$$\begin{aligned} W &= \int (e_1\, i_1 + e_2\, i_2 + \ldots + e_n\, i_n)\, dt \\ &= \int (z_{11}\, i_1^2 + z_{12}\, i_2\, i_1 + \ldots + z_{1n}\, i_n\, i_1)\, dt \\ &\ldots\ldots\ldots\ldots \\ &+ \int (z_{n1}\, i_1\, i_n + z_{n2}\, i_2\, i_n + \ldots + z_{nn}\, i_n^2)\, dt \end{aligned}$$

Then combining the equations $i_{(m+1)}$ to i_n there results

$$\left.\begin{aligned} i_1 &= Y_{11}\, e_1 + \ldots + Y_{m1}\, e_n + Y_{10}\, e_0 \\ &\ldots\ldots\ldots\ldots \\ i_m &= Y_{1m}\, e_1 + \ldots + Y_{mm}\, e_n + Y_{m0}\, e_0 \\ i_0 &= Y_{10}\, e_1 + \ldots + Y_{m0}\, e_n + Y_{00}\, e_0 \end{aligned}\right\} \tag{173}$$

Hereby the number of simultaneous equations has been reduced from n to $(m + 1)$.

The total energy residing in the electrostatic field is

$$\begin{aligned} W_e &= \frac{1}{2}\int (Q_1 e_1 + Q_2 e_2 + \ldots + Q_n e_n)\, dx \\ &= \frac{1}{2}\int (K_{11} e_1^2 + K_{12} e_2 e_1 + \ldots + K_{1n} e_n e_1)\, dx \\ &\ldots\ldots\ldots\ldots \\ &+ \frac{1}{2}\int (K_{n1} e_1 e_n + K_{n2} e_2 e_n + \ldots + K_{nn} e_n^2)\, dx \end{aligned} \tag{174}$$

where the integration is to extend over the lengths of the waves.

The electromagnetic energy (Jeans, p. 443) is

$$\begin{aligned} W_i &= \frac{1}{2}\int (\phi_1\, i_1 + \phi_2\, i_2 + \ldots + \phi_n\, i_n)\, dx \\ &= \frac{1}{2}\int (L_{11}\, i_1^2 + L_{12}\, i_2\, i_1 + \ldots + L_{1n}\, i_n\, i_1)\, dx \\ &\ldots\ldots\ldots\ldots \\ &+ \frac{1}{2}\int (L_{n1}\, i_1\, i_n + L_{n2}\, i_2\, i_n + \ldots + L_{nn}\, i_n^2)\, dx \end{aligned} \tag{175}$$

The total energy of the waves is (considering only the waves moving in one direction),

$$\begin{aligned} W &= \int (e_1\, i_1 + e_2\, i_2 + \ldots + e_n\, i_n)\, dt \\ &= \int (z_{11}\, i_1^2 + z_{12}\, i_2\, i_1 + \ldots + z_{1n}\, i_n\, i_1)\, dt \\ &\ldots\ldots\ldots\ldots \\ &+ \int (z_{n1}\, i_1\, i_n + z_{n2}\, i_2\, i_n + \ldots + z_{nn}\, i_n^2)\, dt \end{aligned}$$

$$= \int (y_{11}\, e_1^2 + y_{12}\, e_2\, e_1 + \ldots + y_{1n}\, e_n\, e_1)\, dt$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \int (y_{n1}\, e_1\, e_n + y_{n2}\, e_2\, e_n + \ldots + y_{nn}\, e_n^2)\, dt \tag{176}$$

But for free traveling waves on overhead lines,

$$z = c\, L \tag{177}$$

and

$$dx = vdt = c\, dt \tag{178}$$

Therefore, Equation (176) may be written

$$W = \int (L_{11}\, i_1^2 + L_{12}\, i_2\, i_1 + \ldots + L_{1n}\, i_n\, i_1)\, dx$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \int (L_{n1}\, i_1\, i_n + L_{n2}\, i_2\, i_1 + \ldots + L_{nn}\, i_n^2)\, dx$$

$$= 2\, W_i \tag{179}$$

thus proving that the energy of the system is divided equally between the electrostatic and magnetic fields for free waves traveling in one direction. While waves moving in opposite directions are passing through each other the total energy is not equally divided, but may be distributed in any proportion between the two fields. This is also true at a transition point, where the incident waves give rise to reflected waves. In such cases the energies must be computed from Equations (174) and (175) and added to find the total. Equation (176) applies only to waves moving in the same direction, and although it serves to determine the total energy by computing the energies in each system of oppositely moving waves and adding them, it does not hold for resultant potentials and currents.

Case II. The Completely Transposed Line.—If all n conductors are completely transposed with respect to each other and to the ground, and if the conductors have the same resistance, then in effect

$$L_{rr} = L, \qquad K_{rr} = K, \qquad G_{rr} = G$$

$$L_{rs} = L', \qquad K_{rs} = K', \qquad G_{rs} = G'$$

$$Z_{rr}\, Y_{rr} = (R + p\, L)\,(G + p\, K) = Z\, Y$$

$$Z_{rr}\, Y_{rs} = (R + p\, L)\,(G' + p\, K') = Z\, Y'$$

$$Z_{rs}\, Y_{rr} = p\, L'\,(G + p\, K) = Z'\, Y$$

$$Z_{rs}\, Y_{rs} = p\, L'\,(G' + p\, K') = Z'\, Y'$$

$$\begin{aligned} J_{rs} &= Z_{1r}\,Y_{1s} + Z_{2r}\,Y_{2s} + \ldots + Z_{rr}\,Y_{rs} \\ &\quad + \ldots + Z_{sr}\,Y_{ss} + \ldots + Z_{nr}\,Y_{ns} \\ &= Z\,Y' + Z'\,Y + (n-2)\,Z'\,Y' = J \end{aligned} \tag{180}$$

$$\begin{aligned} J_{rr} &= Z_{1r}\,Y_{1r} + Z_{2r}\,Y_{2r} + \ldots + Z_{rr}\,Y_{rr} \\ &\quad + \ldots + Z_{nr}\,Y_{nr} \\ &= Z\,Y + (n-1)\,Z'\,Y' \end{aligned} \tag{181}$$

$$A_{rr} = \left(J_{rr} - \frac{\partial^2}{\partial x^2}\right) = \left[Z\,Y + (n-1)\,Z'\,Y' - \frac{\partial^2}{\partial x^2}\right] = A \tag{182}$$

$$(A - J) = \left[(Z - Z')\,(Y - Y') - \frac{\partial^2}{\partial x^2}\right] \tag{183}$$

Hereby, upon dividing each column through by J and calling $A/J = a$, subtracting adjacent columns from each other, dividing out $(a - 1)$, adding all rows to the last, and finally expanding the remaining determinate in terms of the minors of which the lower right-hand elements are the cofactors, the determinate of Equation (137) becomes

$$\begin{vmatrix} A & J & . & . & . & J \\ J & A & . & . & . & J \\ . & . & . & . & . & . \\ J & J & . & . & . & A \end{vmatrix} e = (A - J)^{n-1}\,(A - J + n\,J)\,e = 0 \tag{184}$$

This is the operational equation of the completely transposed line in terms of the operators A and J. Upon substituting the values of A and J from Equations (180) and (182) it takes the form

$$\left(v_1{}^2\,p^2 + w_1\,p + u_1 - \frac{\partial^2}{\partial x^2}\right)^{n-1}\left(v_2{}^2\,p^2 + w_2\,p + u_2 - \frac{\partial^2}{\partial x^2}\right)e = 0 \tag{185}$$

If the line is free of losses, as well as completely transposed, then (184) becomes

$$\left(v_1{}^2\,p^2 - \frac{\partial^2}{\partial x^2}\right)^{n-1}\left(v_2{}^2\,p^2 - \frac{\partial^2}{\partial x^2}\right)e = 0 \tag{186}$$

This is satisfied by the equation of wave motion

$$e = f\,(x + vt) \tag{187}$$

which substituted in (186) gives

$$v = \pm\,v_1 \quad \text{and} \quad v = \pm\,v_2 \tag{188}$$

thus showing that there are only two possible velocities of wave propagation on the completely transposed no-loss line. In the case of overhead conductors in air, both of these velocities approach equality with the velocity of light, in agreement with the findings of the previous section.

Case III. Solution for Alternating Currents.—Suppose that the line is operating under steady-state, alternating-current conditions, so that the potentials on the n conductors at coordinate x are given by

$$\left.\begin{aligned} e_1 &= E_1 \sin(\omega t + \theta_1) = \text{imaginary part of } E_1\ \epsilon^{j(\omega t+\theta_1)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ e_n &= E_n \sin(\omega t + \theta_n) = \text{imaginary part of } E_n\ \epsilon^{j(\omega t+\theta_n)} \end{aligned}\right\} \tag{189}$$

where the amplitudes $(E_1 \ldots E_n)$ are functions of x. Substituting (189) in (133) there is

$$\frac{\partial^2 \dot{E}_1}{\partial x^2} = \dot{J}_{11}\,\dot{E}_1 + \dot{J}_{12}\,\dot{E}_2 + \ldots + \dot{J}_{1n}\,\dot{E}_n \tag{190}$$

where

$$\left.\begin{aligned} \dot{E}_1 &= E_1\ \epsilon^{j\theta_1} \\ &\cdots \\ \dot{E}_n &= E_n\ \epsilon^{j\theta_n} \end{aligned}\right\} \tag{191}$$

$$\left.\begin{aligned} \dot{J}_{11} &= J_{11} \text{ when } p = j\,\omega \\ &\cdots\cdots \\ \dot{J}_{1n} &= J_{1n} \text{ when } p = j\,\omega \end{aligned}\right\} \tag{192}$$

The common factor $\epsilon^{j\omega t}$ has been canceled out on both sides of Equation (190).

Now according to Equation (137), the general differential equation is the same for every conductor of the n wire system, and therefore each $\dot{E}$ of (190) must follow the same function of x, but the integration constants are, in general, different for each $\dot{E}$. Since the differential equations are ordinary linear differential equations with constant coefficients of order 2 n and homogeneous in $d^2/d\,x^2$ it follows that the solutions are of the form

$$\left.\begin{aligned} \dot{E}_1 &= \sum_1^n (\dot{C}_{1r}\ \epsilon^{\lambda_r x} + \dot{C}_{1r}'\ \epsilon^{-\lambda_r x}) \\ &\cdots\cdots\cdots \\ \dot{E}_n &= \sum_1^n (\dot{C}_{nr}\ \epsilon^{\lambda_r x} + \dot{C}_{nr}'\ \epsilon^{-\lambda_r x}) \end{aligned}\right\} \tag{193}$$

where the $\dot{C}$'s are complex integration constants and the λ's are the roots of Equation (137). There are $2\,n^2$ of these integration constants, of which all except $2\,n$ are redundant. To prove this, substitute Equations (193) into the original equations of type (190) obtaining n equations of the type

$$\begin{aligned}\sum_1^n \lambda_r^2\,(\dot{C}_{1r}\,\epsilon^{\lambda_r x} + \dot{C}_{1r}'\,\epsilon^{-\lambda_r x}) \\ &= \sum_1^n \dot{J}_{11}\,(\dot{C}_{1r}\,\epsilon^{\lambda_r x} + \dot{C}_{1r}'\,\epsilon^{-\lambda_r x}) \\ &\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &\quad + \sum_1^n \dot{J}_{1n}\,(\dot{C}_{nr}\,\epsilon^{\lambda_r x} + \dot{C}_{nr}'\,\epsilon^{-\lambda_r x})\end{aligned} \tag{194}$$

Collecting terms

$$\begin{aligned}\sum_1^n \{[(\lambda_r^2 - \dot{J}_{11})\,\dot{C}_{1r} - \dot{J}_{12}\,\dot{C}_{2r} - \ldots - \dot{J}_{1n}\,\dot{C}_{nr}]\,\epsilon^{\lambda_r x} \\ + [(\lambda_r^2 - \dot{J}_{11})\,\dot{C}_{1r}' - \dot{J}_{12}\,\dot{C}_{2r}' - \ldots - \dot{J}_{1n}\,\dot{C}_{nr}']\,\epsilon^{-\lambda_r x}\} = 0\end{aligned} \tag{195}$$

It is now necessary to digress long enough to prove (what is probably self-evident to many) that each of the coefficients in (195) is individually equal to zero. In the general case let

$$A\,f_1\,(x) + B\,f_2\,(x) + \ldots + N\,f_n\,(x) = 0 \tag{196}$$

where the functions $f\,(x)$ are all different. Assuming that each of these admits of expansion as a power series in x, by Maclaurin's theorem, there is

$$\begin{aligned}A\,(a_1 + b_1\,x + c_1\,x^2 + \ldots) + \ldots \\ + N\,(a_n + b_n\,x + c_n\,x^2 + \ldots) = 0\end{aligned} \tag{197}$$

Collecting terms

$$\begin{aligned}(a_1\,A + a_2\,B + \ldots + a_n\,N) \\ + (b_1\,A + b_2\,B + \ldots + b_n\,N)\,x + \ldots = 0\end{aligned} \tag{198}$$

But by the method of indeterminate coefficients, each term of this power series in x must individually equal zero, so that there are the simultaneous equations

$$\left.\begin{aligned} a_1\,A + a_2\,B + \ldots + a_n\,N = 0 \\ b_1\,A + b_2\,B + \ldots + b_n\,N = 0 \\ c_1\,A + c_2\,B + \ldots + c_n\,N = 0 \end{aligned}\right\} \tag{199}$$

etc.

Since the coefficients ($a, b, c \ldots$) are entirely arbitrary, the solution of these equations leads to a determinate of which the denominator is finite and the numerator is zero (by virtue of a column of zeros), and therefore

$$A = B = \ldots = N = 0 \tag{200}$$

Thus in any equation of type (196) the individual coefficients are separately equal to zero.

Returning now to the equations of type (195), and considering all n of these equations, there is

$$\left.\begin{aligned}
(\dot{J}_{11} - \lambda_r^2)\,\dot{C}_{1r} + \dot{J}_{12}\,\dot{C}_{2r} + \ldots + \dot{J}_{1n}\,\dot{C}_{nr} &= 0 \\
\dot{J}_{21}\,\dot{C}_{1r} + (\dot{J}_{22} - \lambda_r^2)\,\dot{C}_{2r} + \ldots + \dot{J}_{2n}\,\dot{C}_{nr} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{J}_{n1}\,\dot{C}_{1r} + \dot{J}_{n2}\,\dot{C}_{2r} + \ldots + (\dot{J}_{nn} - \lambda_r^2)\,\dot{C}_{nr} &= 0
\end{aligned}\right\} \tag{201}$$

and exactly the same relationships hold between the $\dot{C}'$ coefficients. Now in order that (201) may be satisfied by values of the $\dot{C}$'s other than zero, the denominator of the determinate must be equal to zero, that is

$$\begin{vmatrix}
(\dot{J}_{11} - \lambda_r^2) & \dot{J}_{12} & \ldots & \dot{J}_{1n} \\
\dot{J}_{21} & (\dot{J}_{22} - \lambda_r^2) & \ldots & \dot{J}_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\dot{J}_{n1} & \dot{J}_{n2} & \ldots & (\dot{J}_{nn} - \lambda_r^2)
\end{vmatrix} = 0 \tag{202}$$

Therefore, if (202) holds, there are $(n - 1)$ independent relationships between the $\dot{C}$'s in Equation (201), so that any $(n - 1)$ of them may be eliminated. But since there are n values of r, there will remain n integration constants that must be determined from the terminal conditions. Likewise, there will remain n arbitrary integration constants among the $\dot{C}'$ coefficients.

Thus, the n-wire transmission system has associated with it n propagation constants λ_r and $2\,n$ integration constants $\dot{C}_r$ and $\dot{C}'_r$.

If the line is a completely transposed three-wire line, there is by Equations (184), (180), and (182)

$$\left[(Z - Z')\,(Y - Y') - \frac{\partial^2}{\partial x^2}\right]^2$$

$$\left[(Z + 2\,Z')\,(Y + 2\,Y') - \frac{\partial^2}{\partial x^2}\right]\dot{E} = 0 \tag{203}$$

Therefore the propagation constants are

$$\left.\begin{aligned} \lambda_1^2 &= (Z - Z')\,(Y - Y') \\ \lambda_2^2 &= (Z + 2\,Z')\,(Y + 2\,Y') \end{aligned}\right\} \tag{204}$$

Substituting λ_1^2 in (201) there results

$$\dot{C}_{11} + \dot{C}_{21} + \dot{C}_{31} = 0 \tag{205}$$

Hence

$$\dot{C}_{11} = -\,(\dot{C}_{21} + \dot{C}_{31}) \tag{206}$$

Likewise

$$\dot{C}_{11}' = -\,(\dot{C}_{21}' + \dot{C}_{31}') \tag{207}$$

Substituting λ_2^2 in (201) there results

$$\dot{C}_{12} = \dot{C}_{22} = \dot{C}_{32} \tag{208}$$

$$\dot{C}_{12}' = \dot{C}_{22}' = \dot{C}_{32}' \tag{209}$$

Therefore the solution for a completely transposed three-wire line is

$$\left.\begin{aligned} \dot{E}_1 &= \dot{C}_{11}\,\epsilon^{\lambda_1 x} + \dot{C}_{11}'\,\epsilon^{-\lambda_1 x} + \dot{C}_{12}\,\epsilon^{\lambda_2 x} + \dot{C}_{12}'\,\epsilon^{-\lambda_2 x} \\ \dot{E}_2 &= \dot{C}_{21}\,\epsilon^{\lambda_1 x} + \dot{C}_{21}'\,\epsilon^{-\lambda_1 x} + \dot{C}_{12}\,\epsilon^{\lambda_2 x} + \dot{C}_{12}'\,\epsilon^{-\lambda_2 x} \\ \dot{E}_3 &= \dot{C}_{31}\,\epsilon^{\lambda_1 x} + \dot{C}_{31}'\,\epsilon^{-\lambda_1 x} + \dot{C}_{12}\,\epsilon^{\lambda_2 x} + \dot{C}_{12}'\,\epsilon^{-\lambda_2 x} \end{aligned}\right\} \tag{210}$$

and $\dot{C}_{11}$ and $\dot{C}_{11}'$ are given by (206) and (207). There are thus six independent integration constants that must be determined from the terminal conditions. If the system is a balanced three-phase circuit (no zero sequence components), then $\dot{C}_{12} = 0$ and $\dot{C}_{12}' = 0$ and (210) reduces to

$$\left.\begin{aligned} \dot{E}_1 &= \dot{C}_{11}\,\epsilon^{\lambda_1 x} + \dot{C}_{11}'\,\epsilon^{-\lambda_1 x} \\ \dot{E}_2 &= \dot{C}_{21}\,\epsilon^{\lambda_1 x} + \dot{C}_{21}'\,\epsilon^{-\lambda_1 x} \\ \dot{E}_3 &= \dot{C}_{31}\,\epsilon^{\lambda_1 x} + \dot{C}_{31}'\,\epsilon^{-\lambda_1 x} \end{aligned}\right\} \tag{210}$$

If for the complex number λ_1 there be substituted

$$\lambda = \alpha + j\,\beta \tag{211}$$

then the equations of (210) may be expressed in any of the following familiar forms:

$$\begin{aligned} \dot{E} &= A\,\epsilon^{\lambda x} + B\,\epsilon^{-\lambda x} \\ &= A\,\epsilon^{\alpha x}\,(\cos\beta\,x + j\sin\beta\,x) + B\,\epsilon^{-\alpha x}\,(\cos\beta\,x - j\sin\beta\,x) \end{aligned}$$

$$= (A + B) \cosh \lambda x + (A - B) \sinh \lambda x$$

$$= (A + B) (\cosh \alpha x \cdot \cos \beta x + j \sinh \alpha x \cdot \sin \beta x)$$

$$+ (A - B) (\sinh \alpha x \cdot \cos \beta x + j \cosh \alpha x \cdot \sin \beta x) \qquad (212)$$

This is the so-called vector solution. The actual potential as function of x and t is

$$e = \text{imaginary part of } \dot{E}\ \epsilon^{j\omega t}$$

$$= \text{imaginary part of } (A\ \epsilon^{\lambda x} + B\ \epsilon^{-\lambda x})\ \epsilon^{j\omega t} \qquad (213)$$

It is worth noticing from (204) that λ_1 is in terms of the so-called "constants to neutral" used in practical transmission-line calculations. For if

h = geometric mean height above ground
s = geometric mean spacing between conductors
ρ = radius of conductors

$$Z = R + j\left(\frac{1}{2} + 2 \log \frac{2h}{\rho}\right)10^{-9}$$

$$Z' = + j\left(2 \log \frac{2h}{s}\right)10^{-9}$$

$$(Z - Z') = R + j\left(\frac{1}{2} + 2 \log \frac{s}{\rho}\right)10^{-9} \text{ ohm}$$

$$(Y - Y') = (G - G') + \frac{j}{\left(18 \log \frac{s}{\rho}\right)10^{-11}}$$

From the above discussion and derivations it is apparent that the conventional transmission theory is based on the following assumptions:

1. Completely transposed conductors.
2. Balanced and symmetrical voltages.

SUMMARY OF CHAPTER VI

Traveling waves on the wires of a multi-conductor system react upon each other and therefore depend not only on the self surge impedance of each conductor, as in single-circuit theory, but also on the mutual surge impedances, or coupling, between conductors. These surge impedances are defined in terms of Maxwell's electrostatic and electromagnetic coefficients. The voltage and current waves are related by the system of simultaneous equations

$$\left.\begin{aligned} \pm e_1 &= Z_{11}\, i_1 + \ldots + Z_{n1}\, i_n \\ &\cdots\cdots\cdots\cdots \\ \pm e_n &= Z_{1n}\, i_1 + \ldots + Z_{nn}\, i_n \end{aligned}\right\}$$

$$\left.\begin{aligned} \pm i_1 &= Y_{11}\, e_1 + \ldots + Y_{n1}\, e_n \\ &\cdots\cdots\cdots\cdots \\ \pm i_n &= Y_{1n}\, e_1 + \ldots + Y_{nn}\, e_n \end{aligned}\right\}$$

where the plus sign is used for waves traveling in the forward direction and the minus sign for waves traveling in the reverse direction. The Z and Y coefficients (surge impedances and admittances, respectively) are related by these two systems of simultaneous linear equations, as shown in the text, Equations (159) to (162), and the surge impedances may be calculated from Equations (157).

Theoretically, there are as many velocities of propagation on a multi-conductor system as there are conductors, but on overhead lines these all become equal to the velocity of light.

Under certain conditions, when a group of conductors on a multi-conductor system is constrained to carry equipotential waves, it is convenient to replace this group of conductors by an equivalent single conductor having the same potential, the same total current, and the same external effect on adjacent conductors not of the group. The transformation is given in Equations (165) to (170) and (173).

Energy relationships of the waves on a multi-conductor system have been derived in Equations (174) to (179). It is shown that the energy is half electrostatic and half electromagnetic for free traveling waves.

When the line losses are included, the differential equations of the multi-conductor system become very complicated. Great simplification results when the line is completely transposed, that is, when every conductor occupies the same relative position with respect to the other conductors and the ground plane for the same distance.

The steady-state, alternating-current solution for the n-wire transmission system involves n propagation constants and $2\,n$ integration constants. If the line is completely transposed the propagation constants reduce to two, and if the system is balanced one of these propagation constants vanishes and the other involves the so-called "constants to neutral" used in conventional transmission-line calculations.

CHAPTER VII

TRANSITION POINTS OF THE MULTI-CONDUCTOR CIRCUIT*

Although there is no limit to the complexity of the impedance network at a transition point on the multi-conductor circuit, yet for most practical cases that shown in Fig. 53 is sufficiently general. Indeed, the procedure followed in setting up and solving the equations for the reflected and refracted waves, as well as the currents and voltages in all branches of any transition point network, is the same, so that the method of solution which will be given applies generally.

Referring to Fig. 53, let

$Y_{11}, Y_{22}, \ldots, Y_{nn}$ = self surge admittances of lines on the left.

Y_{12}, Y_{13}, etc., = mutual surge admittances of lines on the left.

$y_{11}, y_{22}, \ldots, y_{nn}$ = self surge admittances of lines on the right.

y_{12}, y_{13}, etc., = mutual surge admittances of lines on the right.

$U_1, U_2, \ldots, U_n$ = series impedance network on the left.

$W_1, W_2, \ldots, W_n$ = series impedance network on the right.

$N_1, N_2, \ldots, N_n$ = admittances to ground.

N_{12}, N_{23}, etc., = admittances from junction to junction.

e, i = potential and current incident waves.

e', i' = potential and current reflected waves.

e'', i'' = potential and current transmitted waves.

When the incident waves arrive at the transition points, they give rise to reflected and transmitted waves which satisfy the general equations of the transmission line, and are in accord with Kirchhoff's

* "Critique of Ground Wire Theory," by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 49. "Traveling Waves on Transmission Systems," by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 50.

laws and the conditions of current and voltage continuity at the junctions.

The total potential at the junction on any incoming line r is the sum of the incident and reflected waves on that line

$$(e_r + e_r') \tag{214}$$

and the total current is, by (154)

$$(i_r + i_r') = Y_{r1}(e_1 - e_1') + \ldots + Y_{rn}(e_n - e_n') \tag{215}$$

The potential across the admittance N_r is

$$E_r = (e_r + e_r') - U_r(i_r + i_r') \tag{216}$$

and the current through N_r therefore is

$$I_r = N_r E_r \tag{217}$$

The current transmitted to the outgoing line is

$$i_r'' = y_{r1} e_1'' + y_{r2} e_2'' + \ldots + y_{rn} e_n'' \tag{218}$$

and the current transferred to the other junctions is

$$I_r' = N_{1r}(E_r - E_1) + N_{2r}(E_r - E_2) + \ldots + N_{nr}(E_r - E_n) \tag{219}$$

The condition of current continuity requires that

$$i_r + i_r' = i_r'' + I_r + I_r' \tag{220}$$

The potential wave transmitted to the outgoing line is

$$e_r'' = E_r - W_r i_r'' \tag{221}$$

Substituting (215), (216), (217), (218), and (219) in (220), and rearranging, there is

$$\begin{aligned}
&[Y_{r1} + Y_{r1} U_r (N_r + N_{1r} + \ldots + N_{nr}) - Y_{11} N_{1r} U_1 \\
&\qquad - \ldots - Y_{n1} N_{nr} U_n](e_1 - e_1') \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&+ [Y_{rn} + Y_{rn} U_r (N_r + N_{1r} + \ldots + N_{nr}) - Y_{1n} N_{1r} U_1 \\
&\qquad - \ldots - Y_{nn} N_{nr} U_n](e_n - e_n') \\
&+ N_{1r}(e_1 + e_1') + \ldots + N_{nr}(e_n + e_n') \\
&\qquad - (N_r + N_{1r} + \ldots + N_{nr})(e_r + e_r') \\
&= (y_{r1} e_1'' + \ldots + y_{rn} e_n'')
\end{aligned} \tag{222}$$

Substituting (215), (216), (217), (218), and (219) in (221), and rearranging, there is

$$(e_r + e_r') - U_r [Y_{r1} (e_1 - e_1') + \ldots + Y_{rn} (e_n - e_n')]$$
$$= e_r'' + W_r (y_{r1} e_1'' + \ldots + y_{rn} e_n'') \tag{223}$$

For an n-wire system, n equations of type (222) and n equations of type (223) can be written, and these $2n$ simultaneous equations suffice for the determination of the $2n$ unknowns $(e_1' \ldots e_n', e_1'' \ldots e_n'')$. The other quantities may then be found from Equations (214) to (221). These equations are therefore sufficient to formulate completely the behavior of the incident, reflected, and transmitted waves at a general transition point. Some simplifications and examples are given below.

Mutual Connecting Networks Removed.—Suppose that N_{12}, N_{23}, etc., are all zero. Then Equations (222) and (223) reduce to

$$(1 + N_r U_r)[Y_{r1}(e_1 - e_1') + \ldots + Y_{rn}(e_n - e_n')] - N_r(e_r + e_r')$$
$$= (y_{r1} e_1'' + \ldots + y_{rn} e_n'') \tag{224}$$

$$(e_r + e_r') - U_r [Y_{r1} (e_1 - e_1') + \ldots + Y_{rn} (e_n - e_n')]$$
$$= e_r'' + W_r (y_{r1} e_1'' + \ldots + y_{rn} e_n'') \tag{225}$$

Single-Wire Line.—In this case only e_1, e_1', and e_1'' exist, and equations (224) and (225) become

$$(1 + N_1 U_1) Y_{11} (e_1 - e_1') - N_1 (e_1 + e_1') = y_{11} e_1'' \tag{226}$$

$$- U_1 Y_{11} (e_1 - e_1') + (e + e_1') = (1 + W_1 y_{11}) e_1'' \tag{227}$$

Solving these two simultaneous equations for the reflected and transmitted waves, substituting $Z_{11} = 1/Y_{11}$ and $z_{11} = 1/y_{11}$, and dropping subscripts, there is

$$e' = \frac{(z + W)(1 + NU) + U - Z - ZN(z + W)}{(z + W)(1 + NU) + U + Z + ZN(z + W)} e \tag{228}$$

$$e'' = \frac{2z}{(z + W)(1 + NU) + U + Z + ZN(z + W)} e \tag{229}$$

The conventional traveling wave theory is based on a single-wire line and is expressed by the above equations. In terms of the total impedance at the transition point,

$$Z_0 = U + \frac{1}{\left(N + \dfrac{1}{W + z}\right)} = \frac{U + (1 + NU)(W + z)}{1 + N(W + z)} \tag{230}$$

the above equations take the more familiar forms derived in Chapter I.

$$e' = \frac{Z_0 - Z}{Z_0 + Z} e \tag{231}$$

$$e'' = \frac{1}{1 + N\,(W + z)} \frac{2\,Z}{Z_0 + Z} e \tag{232}$$

Energy Relationships at the Junctions.—The energy of a free traveling wave is given by Equations (174) to (179). During the time that the incident waves are at the junction, a redistribution of energy is taking place. The division of energy during this transition period furnishes a valid check on the reflection, refraction, and transfer operators, and is of interest on its own account. At any time t, counting from the instant when the system of incident waves $(e_1 \ldots e_n)$ arrives at the junction, there is

$$\int_t^\infty (e_1\,i_1 + \ldots + e_n\,i_n)\,dt$$
$$= \text{energy remaining in the incident waves} \tag{233}$$

$$\int_0^t (e_1'\,i_1' + \ldots + e_n'\,i_n')\,dt$$
$$= \text{energy in the reflected waves} \tag{234}$$

$$\int_0^t (e_1''\,i_1'' + \ldots + e_n''\,i_n'')\,dt$$
$$= \text{energy in the transmitted waves} \tag{235}$$

$$\int_0^t (E_1\,I_1 + \ldots + E_n\,I_n)\,dt$$
$$= \text{energy absorbed by the networks } (N_1 \ldots N_n) \tag{236}$$

$$\int_0^t [(e_1 + e_1' - E_1)\,(i_1' + i_1) + \ldots + (e_n + e_n' - E_n)\,(i_n' + i_n)]\,dt$$
$$= \text{energy absorbed by the networks } (U_1 \ldots U_n) \tag{237}$$

$$\int_0^t [(E_1 - e_1'')\,i_1'' + \ldots + (E_n - e_n'')\,i_n]\,dt$$
$$= \text{energy absorbed by the networks } (W_1 \ldots W_n) \tag{238}$$

$$\int_0^t \sum_1^n \sum_{s=1} (E_r - E_s)\cdot N_{rs}\,(E_r - E_s)\,dt$$
$$= \text{energy absorbed in connecting networks } N_{rs} \tag{239}$$

The $\sum_s$ summation in (239) ranges from $s = 1$ to $s = n$, excepting $s = r$.

Equating the sum of these energies to the energy in the original incident waves, by the conservation of energy,

$$\int_0^\infty \sum_1^n e_r\, i_r\, dt = \int_t^\infty \sum_1^n e_r\, i_r\, dt - \int_0^t \sum_1^n e_r'\, i_r'\, dt$$

$$+ \int_0^t \sum_1^n e_r''\, i_r''\, dt + \int_0^t \sum_1^n E_r\, I_r\, dt$$

$$+ \int_0^t \sum_1^n (e_r + e_r' - E_r)\, (i_r + i_r')\, dt$$

$$+ \int_0^t \sum_1^n (E_r - e_r'')\, i_r''\, dt$$

$$+ \int_0^t \sum_1^n \sum_{s=1} (E_r - E_s) \cdot N_{rs}\, (E_r - E_s)\, dt \quad (240)$$

Combining the first term on the right with the term on the left, according to the rule

$$\int_0^\infty - \int_t^\infty = \int_0^t$$

and discarding the integrals, there is

$$\sum_{r=1}^n [e_r\, i_r + e_r'\, i_r' - e_r''\, i_r'' - E_r\, I_r - (e_r + e_r' - E_r)\, (i_r + i_r')$$

$$- (E_r - e_r'')\, i_r'' - \sum_s (E_r - E_s) \cdot N_{rs}\, (E_r - E_s)] = 0 \quad (241)$$

The currents and voltages at a transition, as determined by the reflection, refraction, and transfer operators, must satisfy Equation (241).

Special Cases.—In the following examples the application of the general equations derived above is restricted to two-wire circuits, since these most simple multi-conductor circuits adequately illustrate the methods of analysis with a minimum amount of algebraic exercise. Increasing the number of conductors involved merely magnifies the amount of algebra that must be done, without serving any other useful purpose. In the chapter on ground wires an n-conductor transition point problem is worked out in detail; and in the chapter on component kinds of waves, a six-conductor circuit is completely solved.

When some of the transition point networks are zero and others are infinite, then the general equations may become indeterminate. Recourse may then be had to one or the other of two procedures. Either evaluate the indeterminate by substituting symbols for the elements giving rise to the indeterminate and then eliminating the troublesome terms between equations before allowing them to take on their limiting values; or else set up the equations from the beginning in the same way that (222) and (223) were established. This latter procedure is by far the safer. Consider, for example, the case of Fig. 55, where two incoming lines are bussed together at the transition point and there is only one outgoing line. Then

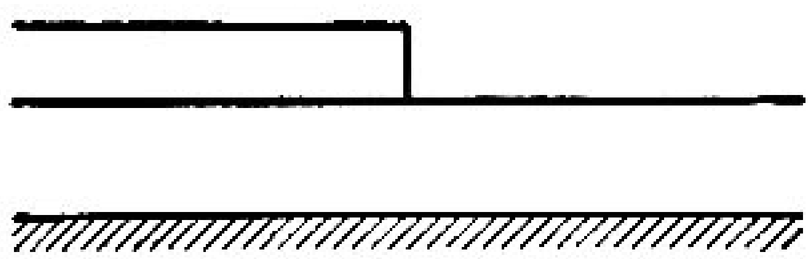


FIG. 55.—Two Lines Bussed

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$N_{12} = \infty$$

$$y_{11} = y_{12} = 0$$

The general equations (222) and (223) give

$$\begin{aligned}(Y_{11} + Y_{11}\, U_1\, N_{12} - Y_{21}\, N_{12}\, U_2)\,(e_1 - e_1') \\ +(Y_{12} + Y_{12}\, U_1\, N_{12} - Y_{22}\, N_{21}\, U_2)\,(e_2 - e_2') \\ + N_{12}\,(e_2 + e_2') - N_{12}\,(e_1 + e_1') = 0\end{aligned}$$

$$\begin{aligned}(Y_{12} + Y_{12}\, U_2\, N_{12} - Y_{11}\, N_{12}\, U_1)\,(e_1 - e_1') \\ +(Y_{22} + Y_{22}\, U_2\, N_{12} - Y_{12}\, N_{12}\, U_1)\,(e_2 - e_2') \\ + N_{12}\,(e_1 + e_1') - N_{12}\,(e_2 + e_2') = y_{22}\, e_2''\end{aligned}$$

$$e_1 + e_1' = e_1''$$

$$e_2 + e_2' = e_2''$$

Adding the first two equations

$$(Y_{11} + Y_{12})\,(e_1 - e_1') + (Y_{22} + Y_{12})\,(e_2 - e_2') = y_{22}\, e_2''$$

Dividing either of the first two equations through by $N_{12} = \infty$

$$(e_1 + e_1') - (e_2 + e_2') = 0$$

From the four equations immediately above

$$e_1' = \frac{(Y_{11} - Y_{22} - y_{22})\, e_1 + 2\,(Y_{22} + Y_{12})\, e_2}{Y_{11} + Y_{22} + 2\, Y_{12} + y_{22}}$$

$$e_2' = \frac{2\,(Y_{11} + Y_{12})\, e_1 + (Y_{22} - Y_{11} - y_{22})\, e_2}{Y_{11} + Y_{22} + 2\, Y_{12} + y_{22}}$$

$$e_2'' = \frac{2\,(Y_{11} + Y_{12})\, e_1 + 2\,(Y_{22} + Y_{12})\, e_2}{Y_{11} + Y_{22} + 2\, Y_{12} + y_{22}}$$

Of course in a case of this kind much time is saved by writing the transition-point equations directly, rather than reducing from the general equations. In the examples which follow, the general equations are employed, but the reader will find it profitable to work each case out directly. The derivation of the general equations is principally of value in serving as a model for procedure.

Fig. 56*a*. *One of two lines suddenly terminates.*

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$y_{11} = 1/z_{11},\; y_{22} = y_{12} = 0$$

Substituting these values in the general equation there is

$$\left.\begin{aligned}
Y_{11}\,(e_1 - e_1') + Y_{12}\,(e_2 - e_2') &= y_{11}\, e_1'' \\
Y_{21}\,(e_1 - e_1') + Y_{22}\,(e_2 - e_2') &= 0 \\
(e_1 + e_1') &= e_1'' \\
(e_2 + e_2') &= e_2''
\end{aligned}\right\}$$

The solution of these simultaneous equations gives

$$e_1' = \frac{z_{11} - Z_{11}}{z_{11} + Z_{11}}\, e_1$$

$$e_2' = e_2 - \frac{2\, Z_{12}}{z_{11} + Z_{11}}\, e_1$$

$$e_1'' = \frac{2\, z_{11}}{z_{11} + Z_{11}}\, e_1$$

$$Z_{11} = \frac{Y_{22}}{Y_{11}\, Y_{22} - Y_{12}^2}$$

$$Z_{12} = \frac{-\,Y_{12}}{Y_{11}\, Y_{22} - Y_{12}^2}$$

If e_2 was induced on line 2 by e_1 on line 1, then

$$e_2 = \frac{Z_{12}}{Z_{11}} e_1$$

Or conversely, if e_1 was induced by e_2 then

$$e_1 = \frac{Z_{12}}{Z_{22}} e_2$$

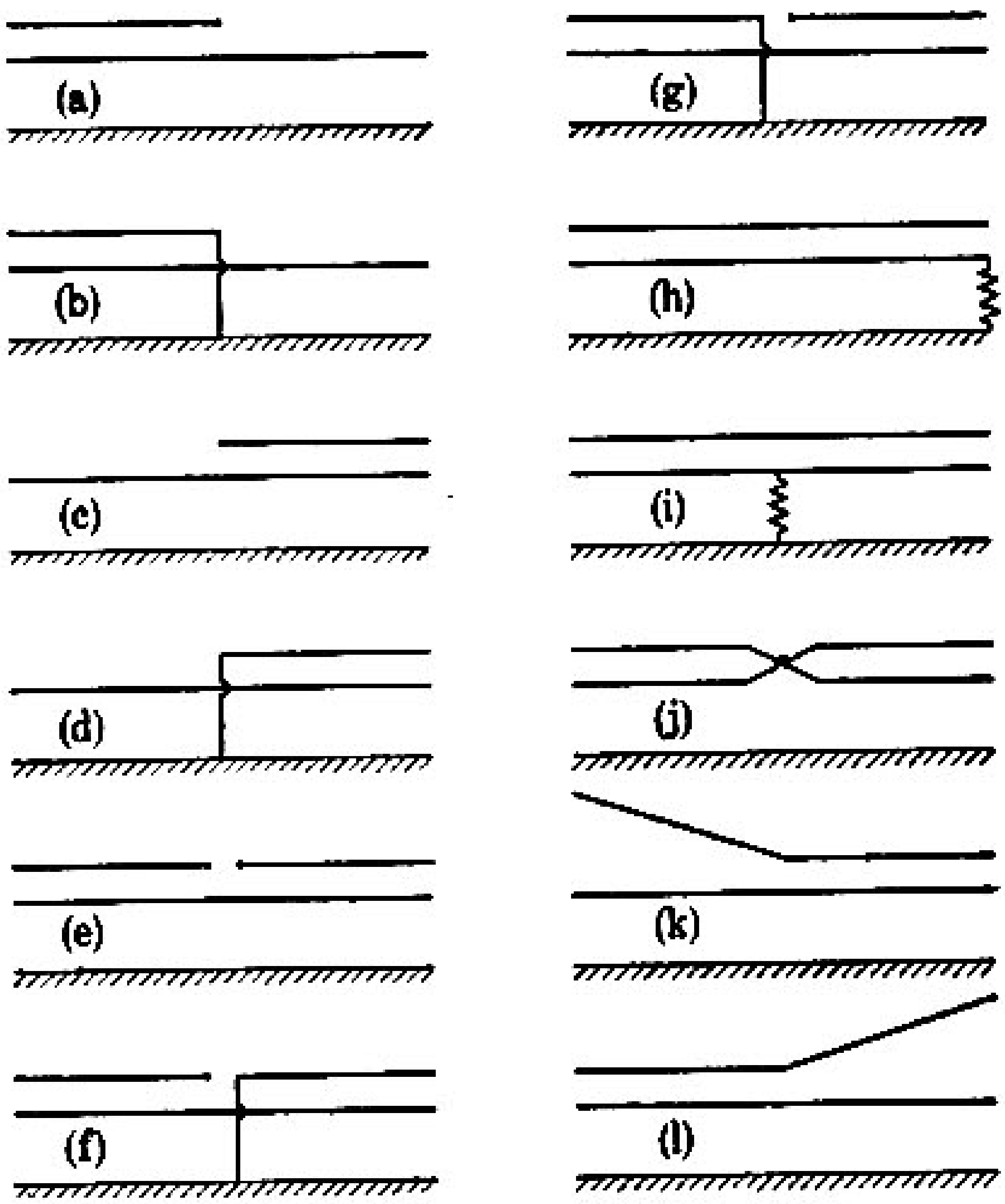


FIG. 56.—Transition Points of a Double Circuit

If, as would likely be the case, the line to the right is simply a continuation of No. 1 wire, then

$$z_{11} = Z_{11}$$

and the equations become

$$e_1' = 0$$

$$e_2' = e_2 - \frac{Z_{12}}{Z_{11}} e_1$$

$$e_1'' = e_1$$

that is, there is no reflection on line 1, and the full wave is transmitted. In this case, had e_2 been induced by e_1 there would be no reflection on No. 2 conductor either.

Fig. 56*b*. *One of two lines is terminated and grounded.*

$$N_1 = U_1 = U_2 = W_1 = W_2 = 0, \quad N_2 = \infty$$

$$y_{11} = 1/z_{11}, \; y_{22} = y_{12} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11}\, e_1'' \\ (e_2 + e_2') &= 0 \\ (e_1 + e_1') \qquad\qquad &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right\}$$

Solving these simultaneous equations there is

$$e_1' = \frac{Y_{11} - y_{11}}{Y_{11} + y_{11}}\, e_1 + \frac{2\, Y_{12}}{Y_{11} + y_{11}}\, e_2$$

$$e_2' = -\, e_2$$

$$e_1'' = \frac{2\, Y_{11}\, e_1}{Y_{11} + y_{11}} + \frac{2\, Y_{12}}{Y_{11} + y_{11}}\, e_2$$

Fig. 56*c*. *Isolated conductor introduced.*

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$Y_{11} = 1/Z_{11}, \; Y_{12} = Y_{22} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') &= y_{11}\, e_1'' + y_{12}\, e_2'' \\ 0 &= y_{21}\, e_1'' + y_{22}\, e_2'' \\ (e_1 + e_1') &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{z_{11} - Z_{11}}{z_{11} + Z_{11}}\, e_1$$

$$e_1'' = \frac{2\, z_{11}}{z_{11} + Z_{11}}\, e_1$$

$$e_2'' = \frac{z_{12}}{z_{11}}\, e_1'' = \frac{2\, z_{12}}{z_{11} + Z_{11}}\, e_1$$

Thus if No. 1 is a through conductor, so that $z_{11} = Z_{11}$, there is no reflection.

Fig. 56d. Grounded conductor intróduced.

$$N_1 = U_1 = U_2 = W_1 = W_2 = 0, \quad N_2 = \infty$$

$$Y_{11} = 1/Z_{11}, \quad Y_{12} = Y_{22} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') &= y_{11} e_1'' + y_{12} e_2'' \\ (e_2 + e_2') &= 0 \\ (e_1 + e_1') &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{Y_{11} - y_{11}}{Y_{11} + y_{11}} e_1$$

$$e_1'' = \frac{2\,Y_{11}}{Y_{11} + y_{11}} e_1$$

$$e_2'' = 0$$

Fig. 56e. Break in one conductor.

$$N_1 = N_2 = U_1 = U_2 = W_1 = 0, \quad W_2 = \infty$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ 0 &= y_{21} e_1'' + y_{22} e_2'' \end{aligned}\right\}$$

Therefore, taking

$$Y_{11} = y_{11}, \quad Y_{22} = y_{22}, \quad \text{and} \quad Y_{12} = Y_{21} = y_{12} = y_{21}$$

$$e_1' = 0$$

$$e_2' = e_2 - \frac{z_{12}}{z_{11}} e_1$$

$$e_1'' = e_1$$

$$e_2'' = \frac{z_{12}}{z_{11}} e_1$$

Fig. 56f. Broken line—far section grounded.

$$U_1 = W_1 = W_2 = N_1 = 0, \quad U_2 = N_2 = \infty$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= 0 \\ (e_1 + e_1') \qquad\qquad\qquad &= e_1'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= 0 \end{aligned}\right\}$$

Therefore, since $e_2'' = 0$

$$e_1' = \frac{1 - Z_{11} y_{11}}{1 + Z_{11} y_{11}} e_1$$

$$e_2' = e_2 + \frac{2 Z_{12} y_{11}}{1 + Z_{11} y_{11}} e_1$$

$$e_1'' = \frac{2}{1 + Z_{11} y_{11}} e_1$$

$$e_2'' = 0$$

Fig. 56g. Broken line—near section grounded.

$$U_1 = U_2 = W_1 = N_1 = 0, \quad W_2 = N_2 = \infty$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ (e_2 + e_2') &= 0 \\ (e_1 + e_1') \qquad\qquad\qquad &= e_1'' \\ 0 &= y_{21} e_1'' + y_{22} e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{z_{11} Y_{11} - 1}{z_{11} Y_{11} + 1} e_1 + \frac{2 Y_{12} z_{11}}{z_{11} Y_{11} + 1} e_2$$

$$e_2' = - e_2$$

$$e_1'' = \frac{2 z_{11}}{z_{11} Y_{11} + 1} (Y_{11} e_1 + Y_{12} e_2)$$

$$e_2'' = \frac{z_{12}}{z_{11}} e_1'' = \frac{2 z_{12}}{z_{11} Y_{11} + 1} (Y_{11} e_1 + Y_{12} e_2)$$

Fig. 56*h*. *One line grounded through a resistor at end of line.*

$$U_1 = U_2 = N_2 = 0, \quad N_1 = 1/R, \quad y_{11} = y_{22} = y_{12} = 0$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') - (e_1 + e_1')/R &= 0 \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= 0 \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{R - Z_{11}}{R + Z_{11}} e_1$$

$$e_2' = e_2 - \frac{2\,Z_{12}}{R + Z_{11}} e_1$$

If $R = Z_{11}$, then $e_1' = 0$ and there is no reflected wave on No. 1 wire. However, a wave

$$e_2' = e_2 - \frac{Z_{12}}{Z_{11}} e_1$$

is reflected on No. 2 wire.

Fig. 56*i*. *Resistance ground on one wire.*

$$U_1 = U_2 = W_1 = W_2 = N_2 = 0, \quad N_1 = 1/R$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' + (e_1 + e_1')/R \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{-\,Z_{11}}{2\,R + Z_{11}} e_1$$

$$e_2' = \frac{-\,Z_{12}}{2\,R + Z_{11}} e_1$$

$$e_1'' = \frac{2\,R}{2\,R + Z_{11}} e_1$$

$$e_2'' = e_2 - \frac{Z_{12}}{2\,R + Z_{11}} e_1$$

These equations are of importance in connection with the theory of ground wires.

Fig. 56j. Transposition of a line.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$y_{11} = Y_{22},\ y_{22} = Y_{11},\ y_{12} = y_{21} = Y_{12} = Y_{21}$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right]$$

Therefore

$$e_1' = \frac{(Y_{11} - Y_{22})}{(Y_{11} + Y_{22})^2 - 4\,Y_{12}^2}\left[(Y_{11} + Y_{22})\,e_1 + 2\,Y_{12}\,e_2\right]$$

$$e_2' = \frac{-(Y_{11} - Y_{22})}{(Y_{11} + Y_{22})^2 - 4\,Y_{12}^2}\left[2\,Y_{12}\,e_1 + (Y_{11} + Y_{22})\,e_2\right]$$

$$e_1'' = e_1' + e_1$$

$$e_2'' = e_2' + e_2$$

If the two conductors are in the same horizontal plane so that $Y_{11} = Y_{22}$, then there are no reflections.

If the two incident waves are alike, that is $e_1 = e_2 = e$, then

$$e_1' = -e_2' = \frac{(Y_{11} - Y_{22})\,e}{Y_{11} + Y_{22} - 2\,Y_{12}}$$

Fig. 56k. Line entering a section parallel to another line.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$Y_{12} = 0,\ Z_{11} = z_{11},\ Z_{22} = z_{22}$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') &= y_{11} e_1'' + y_{12} e_2'' \\ Y_{22}(e_2 - e_2') &= y_{21} e_1'' + y_{22} e_2'' \\ (e_1 + e_1') &= e_1'' \\ (e_2 + e_2') &= e_2'' \end{aligned}\right]$$

Therefore

$$e_1' = \frac{[(Y_{11} - y_{11})(Y_{22} + y_{22}) + y_{12}^2]\,e_1 - 2\,y_{12}\,Y_{22}\,e_2}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - y_{12}^2}$$

$$e_2' = \frac{[(Y_{11} + y_{11})(Y_{22} - y_{22}) + y_{12}^2]\,e_2 - 2\,y_{12}\,Y_{11}\,e_1}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - y_{12}^2}$$

$$e_1'' = \frac{2\,Y_{11}(Y_{22} + y_{22})\,e_1 - 2\,y_{12}\,Y_{22}\,e_2}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - y_{12}^2}$$

$$e_2'' = \frac{2\,Y_{22}(Y_{11} + y_{11})\,e_2 - 2\,y_{12}\,Y_{11}\,e_1}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - y_{12}^2}$$

In a case of this kind it is highly improbable that both e_1 and e_2 would exist simultaneously, so that the equation could be simplified to that extent.

Fig. 56l. Line leaving a section parallel to another line.

$$N_1 = N_2 = U_1 = U_2 = W_1 = W_2 = 0$$

$$y_{12} = 0,\ Z_{11} = z_{11},\ Z_{22} = z_{22}$$

$$\left.\begin{aligned} Y_{11}(e_1 - e_1') + Y_{12}(e_2 - e_2') &= y_{11} e_1'' \\ Y_{21}(e_1 - e_1') + Y_{22}(e_2 - e_2') &= y_{22} e_2'' \\ (e_1 + e_1') \qquad\qquad &= \quad e_1'' \\ (e_2 + e_2') &= \quad e_2'' \end{aligned}\right\}$$

Therefore

$$e_1' = \frac{[(Y_{11} - y_{11})(Y_{22} + y_{22}) - Y_{12}^2]\, e_1 + 2\, Y_{12}\, y_{22}\, e_2}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - Y_{12}^2}$$

$$e_2' = \frac{[(Y_{11} + y_{11})(Y_{22} - y_{22}) - Y_{12}^2]\, e_2 + 2\, Y_{12}\, y_{11}\, e_1}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - Y_{12}^2}$$

$$e_1'' = \frac{2\,[Y_{11}(Y_{22} + y_{22}) - Y_{12}^2]\, e_1 + 2\, Y_{12}\, y_{22}\, e_2}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - Y_{12}^2}$$

$$e_2'' = \frac{2\,[Y_{22}(Y_{11} + y_{11}) - Y_{12}^2]\, e_2 + 2\, Y_{12}\, y_{11}\, e_1}{(Y_{11} + y_{11})(Y_{22} + y_{22}) - Y_{12}^2}$$

SUMMARY OF CHAPTER VII

The calculation of the behavior of waves at a transition point on a multi-conductor circuit is straightforward but usually awkward, since it involves the solution of a set of simultaneous equations. In the text the routine procedure is illustrated by deriving the transition-point equations of a general network. These equations define all the reflected and transmitted waves, voltages across all impedance networks, and the currents. The equations are also given for a more simple transition point not involving mutual connecting networks, Equations (224) and (225); and it is then shown that the general equations properly reduce to the reflection and refraction operators of single-conductor theory. By way of illustration, a number of cases are worked out in detail for a two-conductor system, since calculations on this most simple multi-conductor system adequately demonstrate the method of attack and the routine procedure, as well as describe the essential characteristics of reflections and refractions on such systems. Thus the dependence of a wave on one wire upon the experience of waves on adjacent wires is brought out, and it is shown how transition points of different kinds may be identified.

CHAPTER VIII

RESOLUTION OF WAVES INTO COMPONENT KINDS

The introduction of symmetrical components into steady-state, alternating-current analysis reduces unbalanced polyphase circuits to a set of balanced polyphase systems, each of which may be solved as a relatively simple single-phase circuit. It is likewise possible to employ an analogous sort of argument in the theory of traveling waves, whereby the waves on a multi-conductor system may be resolved into a system of components, each of which has a single associated surge impedance and velocity of propagation. Bekku * has shown that the waves on a completely transposed three-phase line may be resolved into two components (Fig. 57), one of which consists of equal waves on all three wires, and the other of waves adding up to zero on the three wires. Satoh † has extended the analysis to the case of two mutually coupled three-phase circuits, each of which is completely transposed with respect to itself and with respect to the other circuit. In that case he shows that there are three kinds of waves, Fig. 58. The first kind consists of the wave between conductors of each circuit. The second kind is the wave between the group of six conductors on one side and the ground on the other. The third is the wave between the group of three conductors of one circuit as one side and the group of three conductors of the other circuit as the other side.

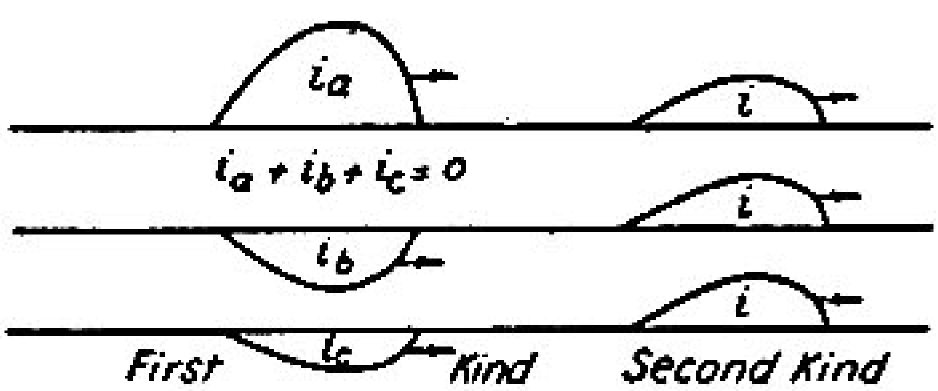


FIG. 57.—Single Circuit Three-Phase Line

In the following derivations and discussion the procedure is generalized.

If $(i_1, i_2, \ldots, i_n)$ comprise a set of current waves having the same

* *Journal* of Japanese Institute of Electrical Engineers, February, 1923.

† *A.I.E.E. Trans.*, 1928.

velocity of propagation v, the corresponding set of potential waves, upon integrating (131) with respect to x, is

$$\left.\begin{array}{l} e_1 = v\,(L_{11}\,i_1 + \ldots + L_{1n}\,i_n) = Z\,i_1 \\ \cdots\cdots\cdots\cdots \\ e_n = v\,(L_{n1}\,i_1 + \ldots + L_{nn}\,i_n) = Z\,i_n \end{array}\right\} \qquad (242)$$

Convérsely, by (132)

$$\left.\begin{array}{l} i_1 = v\,(K_{11}\,e_1 + \ldots + K_{1n}\,e_n) = Y\,e_1 \\ \cdots\cdots\cdots\cdots \\ i_n = v\,(K_{n1}\,e_1 + \ldots + K_{nn}\,e_n) = Y\,e_n \end{array}\right\} \qquad (243)$$

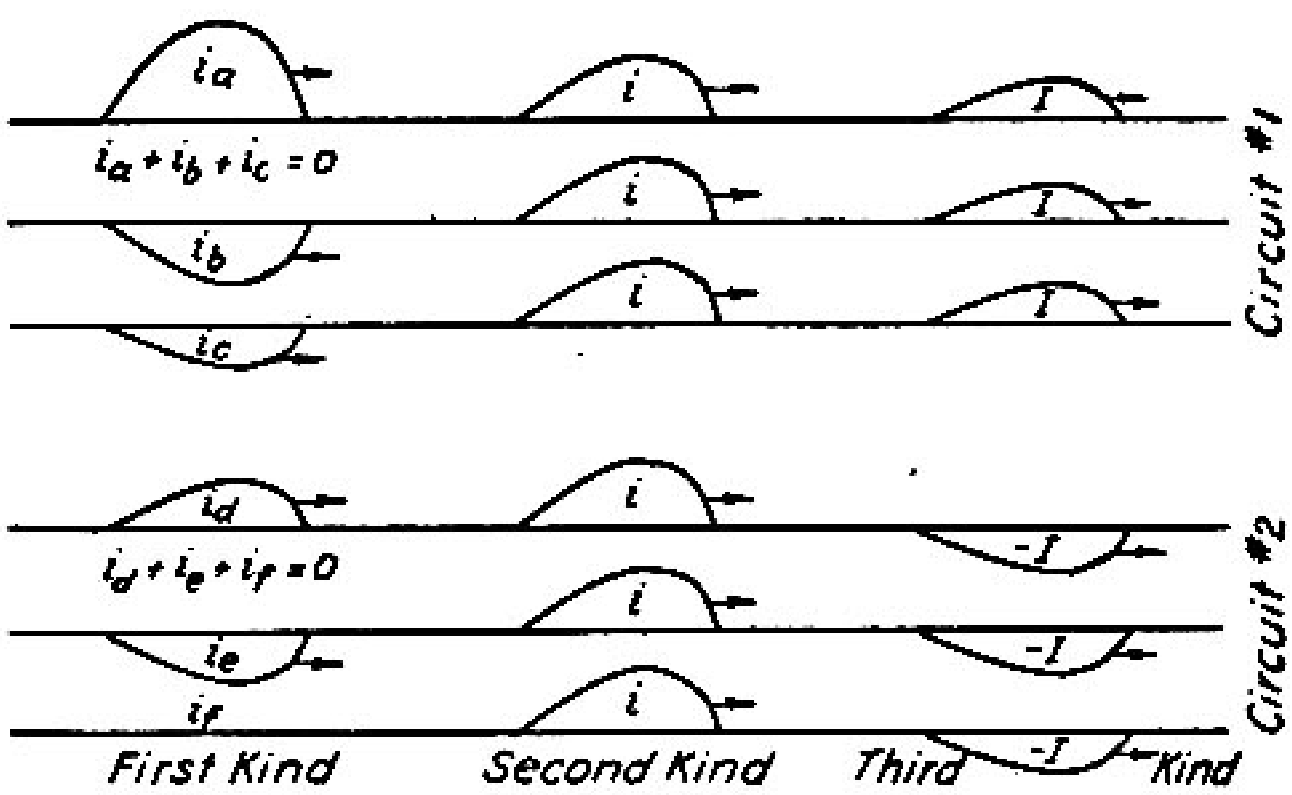


FIG. 58.—Double Circuit Three-Phase Line

in which $Z = 1/Y$ is a coefficient whose existence depends upon finding a solution compatible with (242) and (243). In order that (242) and (243) may have a solution there must be

$$\begin{vmatrix} (L_{11} - Z\,v^{-1}) \ldots & L_{1n} \\ \cdots & \cdots \\ L_{n1} \ldots & (L_{nn} - Z\,v^{-1}) \end{vmatrix} = [(Z\,v^{-1} - a_1) \ldots (Z\,v^{-1} - a_n)] = 0 \qquad (244)$$

$$\begin{vmatrix} (K_{11} - Y\,v^{-1}) \ldots & K_{1n} \\ \cdots & \cdots \\ K_{n1} \ldots & (K_{nn} - Y\,v^{-1}) \end{vmatrix} = [(Y\,v^{-1} - b_1) \ldots (Y\,v^{-1} - b_n)] = 0 \qquad (245)$$

Where the a's and b's are roots of the equations. Therefore

$$\left.\begin{array}{l} Z\,v^{-1} = a \\ Y\,v^{-1} = b \end{array}\right\} \qquad (246)$$

Then, since $Z\,Y = 1$, the velocity is

$$v = \frac{1}{\sqrt{a\,b}} \qquad (247)$$

and the corresponding surge impedance is, by (246)

$$Z = \frac{1}{Y} = a\,v = \sqrt{\frac{a}{b}} \qquad (248)$$

Now as far as the above equations are concerned, any a from (244) may be associated with any b from (245) and therefore yield a corresponding v and Z. However, the necessity of satisfying the general condition, from (144)

$$\begin{vmatrix} B_{11} & \ldots & I_{1n} \\ \cdot & \cdot\ \cdot & \cdot \\ I_{n1} & \ldots & B_{nn} \end{vmatrix} = [(v^{-2} - c_1)\ \ldots\ (v^{-2} - c_n)] = 0 \qquad (249)$$

for the velocities of propagation places a restriction on the pairing of the roots of (244) and (245), so that only those values of a and b may be paired for which v as given by (247) checks a value of v as given by (249). Thus there are just as many *kinds* of waves as there are independent velocities as given by (249).

As a matter of fact, it may be stated without proof that the c coefficients in (249) may always be put in the form

$$c = L\,C$$

where L is an inductance term and C a capacitance term. Then the corresponding surge impedance is

$$Z = \sqrt{L\,C}$$

so that (249) defines both the velocity and the surge impedance of each wave.

Inserting the n values of Z as determined above back into (242), there result n relationships between the currents, some of which may be identical. These relationships define the characteristics of the component waves.

At a transition point the conditions of current and voltage continuity must be satisfied in accordance with Kirchhoff's laws, as applied to the *resultant* currents and voltages of each conductor.

Since these resultant waves are combinations of the component waves, it follows that incident waves of one kind may, in general, cause reflected and refracted waves of all other kinds.

Three-Phase Line.—Consider a three-phase line completely transposed, so that

$$L_{11} = L_{22} = L_{33} = L$$

$$L_{12} = L_{23} = L_{31} = L'$$

$$K_{11} = K_{22} = K_{33} = K$$

$$K_{12} = K_{23} = K_{31} = K'$$

$$I_{11} = I_{22} = I_{33} = L\,K + 2\,L'\,K'$$

$$I_{12} = I_{23} = I_{31} = L\,K' + L'\,K + L'\,K'$$

Then (249) gives

$$(B - I_{12})^2\,(B + 2\,I_{12}) = (I_{11} - I_{12} - v^{-2})^2\,(I_{11} + 2\,I_{12} - v^{-2}) = 0$$

from which
$$v^{-2} = (L - L')\,(K - K')$$

$$v^{-2} = (L + 2\,L')\,(K + 2\,K')$$

From (244) and (245)

$$(L - L' - Z\,v^{-1})^2\,(L + 2\,L' - Z\,v^{-1}) = 0$$

$$(K - K' - Y\,v^{-1})^2\,(K + 2\,K' - Y\,v^{-1}) = 0$$

Therefore by (248)

$$Z_1 = \sqrt{\frac{L - L'}{K - K'}} \quad \textit{for the first kind}$$

$$Z_2 = \sqrt{\frac{L + 2\,L'}{K + 2\,K'}} \quad \textit{for the second kind}$$

Inserting these values back in (242) and solving for the currents:

$$i_1 + i_2 + i_3 = 0 \quad \textit{for the first kind}$$

$$i_1 = i_2 = i_3 = i \quad \textit{for the second kind}$$

Let waves to which Z_2 applies be designated by i and those to which Z_1 applies be designated by (i_a, i_b, i_c).

Then the complete equations are

$$\pm\,e_1 = Z_{11}\,i_1 + Z_{12}\,i_2 + Z_{13}\,i_3 = Z_1\,i_a + Z_2\,i$$

$$\pm\,e_2 = Z_{21}\,i_1 + Z_{22}\,i_2 + Z_{23}\,i_3 = Z_1\,i_a + Z_2\,i$$

$$\pm e_3 = Z_{31} i_1 + Z_{32} i_2 + Z_{33} i_3 = Z_1 i_a + Z_2 i$$

$$i_1 = i + i_a$$

$$i_2 = i + i_b$$

$$i_3 = i + i_c$$

Hence

$$i = \frac{i_1 + i_2 + i_3}{3}$$

$$i_a = i_1 - i$$

$$i_b = i_2 - i$$

$$i_c = i_3 - i$$

Double-Circuit Three-Phase Line.—If the transpositions are such that

$$L_{11} = L_{22} = L_{33} = L_{44} = L_{55} = L_{66} = L$$

$$L_{12} = L_{23} = L_{31} = L_{45} = L_{56} = L_{64} = L'$$

$$L_{14} = L_{15} = L_{16} = L_{24} = L_{25} = L_{26} = L_{34} = L_{35} = L_{36} = L''$$

and likewise for the K's, then

$$I_{11} = I_{22} = I_{33} = I_{44} = I_{55} = I_{66} = (L\,K + 2\,L'\,K' + 3\,L''\,K'')$$

$$I_{12} = I_{23} = I_{31} = I_{45} = I_{56} = I_{64} = (L\,K' + L'\,K + L'\,K' + 3\,L''\,K'')$$

$$I_{14} = I_{15} = I_{16} = I_{24} = I_{25} = I_{26} = I_{34} = I_{35} = I_{36} = (L\,K'' + 2\,L'\,K'' + 2\,L''\,K' + L''\,K)$$

Then (244), (245), and (249) give

$$0 = [v^{-2} - (L - L')\,(K - K')]^4 \, [v^{-2} - (L + 2\,L' - 3\,L'')\,(K + 2\,K' - 3\,K'')] \, [v^{-2} - (L + 2\,L' + 3\,L'')\,(K + 2\,K' + 3\,K'')]$$

$$Z_1 = \sqrt{\frac{L - L'}{K - K'}}$$

$$Z_2 = \sqrt{\frac{L + 2\,L' + 3\,L''}{K + 2\,K' + 3\,K''}}$$

$$Z_3 = \sqrt{\frac{L + 2\,L' - 3\,L''}{K + 2\,K' - 3\,K''}}$$

Inserting these values back in (242) and solving for the currents

$$\left.\begin{aligned} i_1 + i_2 + i_3 &= 0 \\ i_4 + i_5 + i_6 &= 0 \end{aligned}\right\} \textit{for the first kind}$$

$$i_1 = i_2 = i_3 = i_4 = i_5 = i_6 = i \quad \textit{for the second kind}$$

$$i_1 = i_2 = i_3 = -i_4 = -i_5 = -i_6 = I \quad \textit{for the third kind}$$

Let the waves to which Z_1 applies be designated by $(i_a, i_b, \ldots, i_f)$; those to which Z_2 applies by (i); and those to which Z_3 applies by (I). Then the complete equations are

$$\begin{aligned}
\pm e_1 &= Z_1 i_a + Z_2 i + Z_3 I = z_{11} i_1 + \ldots + z_{16} i_6 \\
\pm e_2 &= Z_1 i_b + Z_2 i + Z_3 I = z_{21} i_1 + \ldots + z_{26} i_6 \\
\pm e_3 &= Z_1 i_c + Z_2 i + Z_3 I = z_{31} i_1 + \ldots + z_{36} i_6 \\
\pm e_4 &= Z_1 i_d + Z_2 i - Z_3 I = z_{41} i_1 + \ldots + z_{46} i_6 \\
\pm e_5 &= Z_1 i_e + Z_2 i - Z_3 I = z_{51} i_1 + \ldots + z_{56} i_6 \\
\pm e_6 &= Z_1 i_f + Z_2 i - Z_3 I = z_{61} i_1 + \ldots + z_{66} i_6
\end{aligned}$$

$$\begin{aligned}
i_1 &= i_a + i + I \\
i_2 &= i_b + i + I \\
i_3 &= i_c + i + I \\
i_4 &= i_d + i - I \\
i_5 &= i_e + i - I \\
i_6 &= i_f + i - I
\end{aligned}$$

Hence

$$i = \frac{i_1 + i_2 + i_3 + i_4 + i_5 + i_6}{6}$$

$$I = \frac{i_1 + i_2 + i_3 - i_4 - i_5 - i_6}{6}$$

$$\begin{aligned}
i_a &= i_1 - i - I \\
i_b &= i_2 - i - I \\
i_c &= i_3 - i - I \\
i_d &= i_4 - i + I \\
i_e &= i_5 - i + I \\
i_f &= i_6 - i + I
\end{aligned}$$

As an example of how waves of one kind may generate waves of all three kinds at a transition point, consider Fig. 59, in which the incident waves are of the second kind. Reflected waves are indicated by primes and refracted waves by double primes. At the break

$$e_1 + e_1' = 0$$

$$i_1'' = 0$$

but for all other conductors

$$e_r + e_r' = e_r''$$

$$i_r + i_r' = i_r''$$

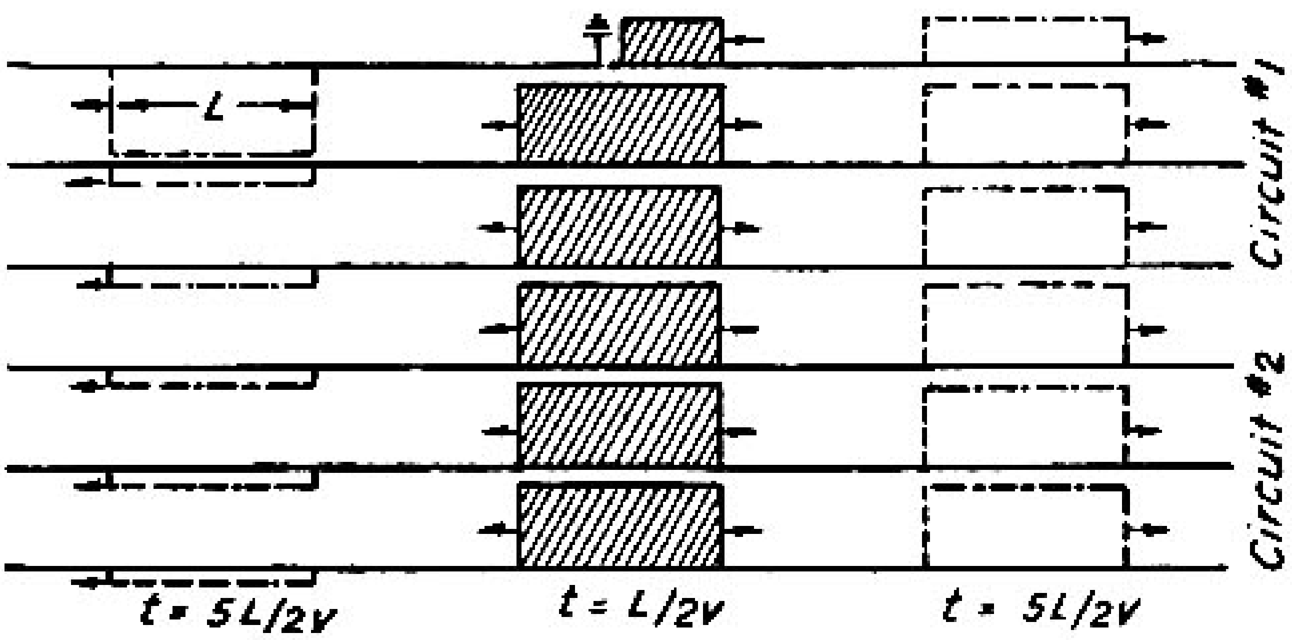


FIG. 59.—Break in One Conductor of a Double-Circuit Three-Phase Line

so that in terms of the three kinds of waves the transition-point equations become

$$\begin{aligned}
Z_2\, i &= Z_1\, i_a' && + Z_2\, i' && + Z_3\, I' \\
Z_2\, i &= Z_1\, (i_b' + i_b'') && + Z_2\, (i' + i'') && + Z_3\, (I' + I'') \\
Z_2\, i &= Z_1\, (i_c' + i_c'') && + Z_2\, (i' + i'') && + Z_3\, (I' + I'') \\
Z_2\, i &= Z_1\, (i_d' + i_d'') && + Z_2\, (i' + i'') && - Z_3\, (I' + I'') \\
Z_2\, i &= Z_1\, (i_e' + i_e'') && + Z_2\, (i' + i'') && - Z_3\, (I' + I'') \\
Z_2\, i &= Z_1\, (i_f' + i_f'') && + Z_2\, (i' + i'') && - Z_3\, (I' + I'') \\
0 &= i_a'' && + i'' && + I'' \\
i &= (i_b'' - i_b') && + (i'' - i') && + (I'' - I') \\
i &= (i_c'' - i_c') && + (i'' - i') && + (I'' - I') \\
i &= (i_d'' - i_d') && + (i'' - i') && - (I'' - I') \\
i &= (i_e'' - i_e') && + (i'' - i') && - (I'' - I') \\
i &= (i_f'' - i_f') && + (i'' - i') && - (I'' - I')
\end{aligned}$$

The currents of the first kind may be eliminated by forming four simultaneous equations as follows: (1) Add Z_1 times the seventh equation to the sum of the first three. (2) Subtract $1/Z_1$ times the first equation from the sum of the third group of three. (3) Add the second group of three. (4) Add the last group of three equations. Then

$$3\,Z_2\,i = 3\,Z_2\,i' + (2\,Z_2 + Z_1)\,i'' + 3\,Z_3\,I' + (2\,Z_3 + Z_1)\,I''$$

$$(Z_2 - 2\,Z_1)\,i = (Z_2 + 2\,Z_1)\,i' - 3\,Z_1\,i'' + (Z_3 + 2\,Z_1)\,I' - 3\,Z_1\,I''$$

$$Z_2\,i = Z_2\,i' + Z_2\,i'' - Z_3\,I' - Z_3\,I''$$

$$i = -\,i' + i'' + I' - I''$$

Herefrom the currents i', i'', I', and I'' may be readily found. Then immediately from the previous equations

$$i_a' = (Z_2\,i - Z_2\,i' - Z_3\,I')/Z_1$$

$$i_a'' = -\,i'' - I''$$

$$i_b' + i_b'' = [Z_2\,i - Z_2\,(i' + i'') - Z_3\,(I' + I'')]/Z_1 = i_c' + i_c''$$

$$i_b' - i_b'' = -\,i + (i'' - i') + (I'' - I') = i_c' - i_c''$$

Hence,

$$i_b' = i_c' = -\frac{i_a'}{2}$$

$$i_b'' = i_c'' = -\frac{i_a''}{2}$$

Likewise

$$i_d' = i_e' = i_f' = 0$$

$$i_d'' = i_e'' = i_f'' = 0$$

Thus in the above case an incident wave of the second kind will generate at the transition point reflected and refracted waves of the first, second, and third kinds on the circuit including the fault, and waves of the second and third kinds on the other circuit.

As a numerical illustration, let $i = 1000$, $Z_1 = 400$, $Z_2 = 1000$, and $Z_3 = 500$. Then the solution of the equations gives

$$
\begin{array}{rlrl}
i_a' = & 1375 & e_a' = & -550{,}000 \\
i_b' = i_c' = & -688 & e_b' = e_c' = & +275{,}000 \\
i' = & 287 & e' = & -287{,}000 \\
I' = & \pm 325 & E' = & \mp 162{,}500 \\
i_a'' = & -617 & e_a'' = & -246{,}800 \\
i_b'' = i_c'' = & 309 & e_b'' = e_c'' = & +123{,}400 \\
i'' = & 790 & e'' = & 790{,}000 \\
I'' = & \mp 173 & E'' = & \mp 86{,}500
\end{array}
$$

These waves have been plotted in Fig. 59.

SUMMARY OF CHAPTER VIII

The waves on a multi-conductor system may be resolved into a number of "kinds" of waves, and to each *kind* of wave there corresponds a single surge impedance. These characteristic surge impedances are defined by the roots of Equation (249), and upon inserting these values back into Equation (242) there result just enough simultaneous equations for the determination of the component waves. When the transmission line is transposed, the resolution into component waves is considerably simplified.

In the case of a single, completely transposed, three-phase line the resolution into kinds gives:

First kind: $i_a + i_b + i_c = 0$
Second kind: $i_1 = i_2 = i_3 = i$

In the case of a double-circuit, three-phase line, each circuit being completely transposed with respect to itself and to the other circuit, the resolution gives:

First kind: $i_a + i_b + i_c = 0,\ i_d = i_e = i_f = 0$
Second kind: $i_1 = i_2 = i_3 = i_4 = i_5 = i_6 = i$
Third kind: $i_1 = i_2 = i_3 = I = -i_4 = -i_5 = -i_6$

Thus the resolution of traveling waves into component kinds is analogous to the resolution of polyphase unbalanced alternating currents into symmetrical components. The resolution greatly simplifies the calculation of the behavior of waves at a transition point. At such a point, waves of one kind may generate reflected waves of other kinds, depending upon the nature of the transition.

CHAPTER IX

TRAVELING WAVES DUE TO LIGHTNING

The formation of thunder clouds and the mechanism of lightning strokes are subjects about which little definite information is available. However, a number of interesting speculations in regard to these phenomena appear to rest on a rational basis. Of these, the theory of G. C. Simpson * has perhaps been the most widely accepted. In fact, most other theories which have been proposed appear to be merely modifications of Simpson's original ideas. He suggested that the rising air currents brushing the falling water drops separate the charges, the positive charges remaining on the water particles while the negative charges are carried by the air to the higher strata of the cloud. The cloud mass is then a good insulator (air) containing a more or less heterogeneous distribution of charge, and these charges are kept separated by mutual repulsion between aggregations. Eventually, however, owing to excessive concentration of charge, or because the cloud approaches the earth, the breakdown gradient of air is exceeded at some local point of rupture, resulting in the formation of an ionized path or streamer. The head of this streamer progresses, establishing the requisite space charge, until new equilibrium conditions are reached; that is, until the gradient has decreased below the breakdown gradient of air. Now as the streamer head advances, it means that a certain part of the initial electrostatic field has been short-circuited and therefore that a higher gradient is available in the neighborhood of the point of rupture, so that the region of ionization is extended further into the interior of the cloud, probably by "preferred paths." Ultimately, then, the whole cloud volume may be permeated with ionized channels, and when the streamer reaches ground all the necessary facilities for a rapid dynamic discharge are thus available. Some writers suppose that there is little or no penetration into the cloud volume until the initial streamer reaches ground, because, they argue, the penetration can not proceed until conditions obtain for increasing the gradient. There is no reason, however, why the short-circuiting of a part of the original electrostatic field by a

* *Phil. Trans. Royal Society*, A-209, 1908.

streamer can not furnish the requisite gradient. It seems necessary to account in some way for the establishment of the principal channels of discharge as a preliminary or incipient stage preceding the dynamic discharge, for otherwise the relatively slow processes of ionization are not comparable with the rapidity of observed discharges—a matter of from ten to a hundred or more microseconds.

Once the main stroke and discharge channels are established the discharge probably is the same as the discharge of any condenser through a resistance. The main stroke may have a length of as much as a mile or more, and is in the nature of a tapered surge impedance. At its upper end the inductance per unit length is larger than at its lower end, while the capacitance per unit length changes in the reverse fashion, if the diameter of the channel is constant, and thus the surge impedance decreases from the cloud to the earth. But branching of the main stroke may more than offset this tapering. It has accordingly become the practice to regard the lightning stroke as a surge impedance of approximately 200 ohms. This assumption will be adhered to in this book, with the understanding that it is most indefinite, but that calculations based upon it have at least a comparative value.

Whether or not the foregoing review of the mechanism of a lightning stroke be right has no bearing on the analysis which is to follow, for the mathematics has been based on observed facts and measurable values.

Let the total charge on the cloud be Q_0 and let $F(t)$ be the law of cloud discharge. If the cloud behaves simply as a capacitor discharging through either a resistance or a surge impedance, then $F(t)$ is exponential. It will be shown, later, that the cathode-ray oscillogram of a lightning surge on a transmission line defines $F(t)$ and that an exponential law is a fair approximation to many typical cases. Then the current in the discharge is

$$I = Q_0 \frac{\partial F(t)}{\partial t} \tag{250}$$

Depending upon the shape and height and size of the cloud the electrostatic field gradient $g(x)$ near the surface of the earth may be specified, and is proportional to the residual charge on the cloud. If the initial field is G, then at any instant t the gradient is

$$g = G\,[1-F(t)] = AQ_0\,[1-F(t)] \tag{251}$$

where A is a function of the height, shape, and size of the cloud and of course can be evaluated only for specific cases.

The potential of the cloud also depends upon its height, shape, size, and charge, that is

$$V = B\, Q_0$$

From the above three equations it is evident that the charge, potential, discharge current, and gradients are related through the law of cloud discharge and the geometric factors of the field.

Traveling Waves Induced by Lightning.—When a charged cloud approaches a transmission line, a charge of opposite sign leaks over the insulators or migrates from the terminals, and appears on the line conductor as a *bound charge* fixed in position by the electrostatic field of the cloud, Fig. 60. The density of bound charge at any point x on the line is proportional to the heights h of the line wires above ground and to the average gradient G between that point and ground, but is modified by the presence of the other conductors and ground

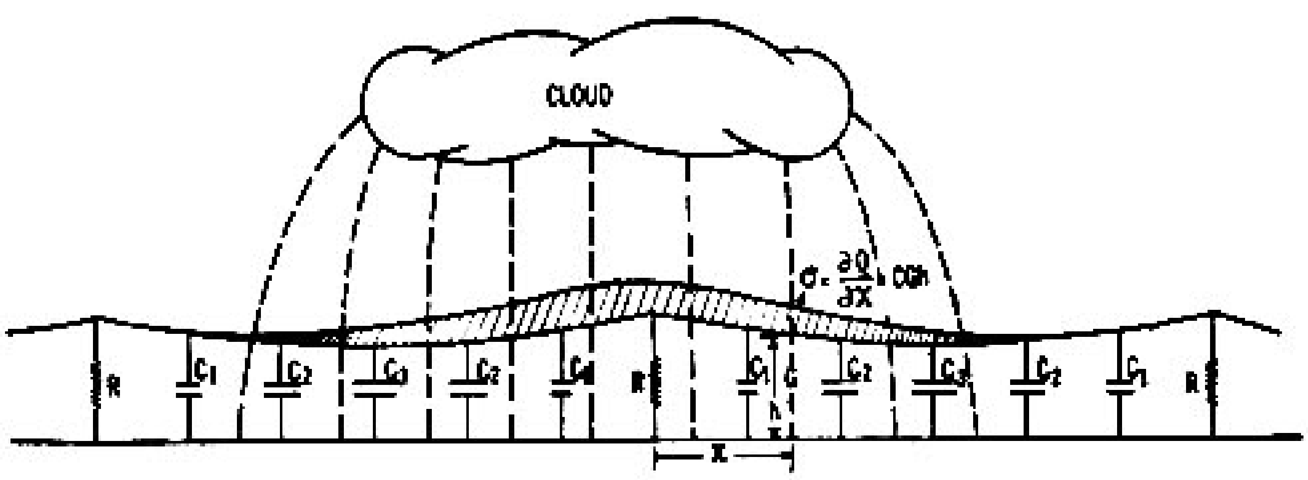


FIG. 60.—Distribution of Bound Charge

wires. At any point x, Maxwell's electrostatic coefficients apply, so that

$$\left.\begin{aligned} V_1 &= 0 = G\,h_1 + p_{11}\,Q_1 + \ldots + p_{1n}\,Q_n \\ V_2 &= 0 = G\,h_2 + p_{21}\,Q_1 + \ldots + p_{2n}\,Q_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V_n &= 0 = G\,h_n + p_{n1}\,Q_1 + \ldots + p_{nn}\,Q_n \end{aligned}\right\} \quad (252)$$

These equations thus give the charges on each conductor before cloud discharge. Since the field of the cloud is not uniform and since the line may sag considerably at midspan, it is evident that the bound charges will not be uniformly distributed, but may have a wide variety of shapes.

Suppose that m of the n wires are ground wires perfectly grounded throughout their length, and let these ideal ground wires be numbered from (1) to (m) inclusive. The ($n - m$) line wires are to be numbered from ($m + 1$) to (n) inclusive. Now if the field gradient G is suddenly removed, the bound charges on the line wires do not change, but those on the ground wires are instantly replaced by new charges Q', for the ground wires remain at zero potential. The equations now become

$$\left.\begin{array}{l}
0 = p_{11}\, Q_1' + \ldots + p_{1m}\, Q_m' + p_{1(m+1)}\, Q_{(m+1)} + \ldots p_{1n}\, Q_n \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
0 = p_{m1}\, Q_1' + \ldots + p_{mm}\, Q_m' + p_{m(m+1)}\, Q_{(m+1)} + \ldots p_{mn}\, Q_n \\
V_{(m+1)} = p_{(m+1)1}\, Q_1' + \ldots + p_{(m+1)m}\, Q_m' \\
\qquad\qquad + p_{(m+1)\,(m+1)}\, Q_{(m+1)} + \ldots p_{(m+1)n}\, Q_n \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
V_n = p_{n1}\, Q_1' + \ldots + p_{nm}\, Q_m' \\
\qquad\qquad + p_{n(m+1)}\, Q_{(m+1)} + \ldots + p_{nn}\, Q_n
\end{array}\right\} \quad (253)$$

Hereby the potentials on the line wires may be determined. Their distribution along the line is the same function of x as that of the initial field gradient G (because Maxwell's equations apply at each point). The released bound charges become traveling waves in accordance with Equations (154) and (156). At the first instant, however, there will be no resultant current flow, so that (154) gives

$$f_k - F_k = 0, \text{ therefore } F_k = f_k \qquad (254)$$

Hence

$$f_k + F_k = 2 f_k = V_k \qquad (255)$$

Therefore, dropping subscripts, the forward and backward traveling waves are identical and are given by

$$f(x) = \tfrac{1}{2} V \qquad (256)$$

But the release of the bound charges is not instantaneous, but according to the law of cloud discharge, $F(t)$. The corresponding traveling waves may then be found by means of Duhamel's theorem

$$e = \int_0^t \{f[x + v(t - \tau)] + f[x - v(t - \tau)]\} \frac{\partial F(\tau)}{\partial \tau} d\tau$$

$$= \lim_{\Delta t \to 0} \sum_{k=0}^{n} \{f[x + v(n - k)\Delta t] + f[x - v(n - k)\Delta t]\} \Delta F_k \qquad (257)$$

where $n \cdot \Delta t = t$, $k \cdot \Delta t = \tau$, and $\Delta F_k = F[(k + 1)\Delta t] - F(k \cdot \Delta t)$. The application of the integral is limited to relatively simple analytic expressions for f and F, but the summation, the limiting case of which is the exact solution, is immediately applicable to any functions f and F, whose graphs are known or assumed. Ultimately, since both these functions must be found from experimental data, it is advisable to deal directly with the summation as an approximation of arbitrary exactness. From it both graphical and tabular methods can be developed. The first two terms under the summation sign represent a pair of forward and backward traveling waves, which are exact replicas of the shape of the traveling waves due to instantaneous cloud discharge, as given by (256) above. The amplitude of these waves is proportional to the increment ΔF in the $F(t)$ curve; and they have moved out from under the original bound charge distribution a total distance $(n - k)\,\Delta t$. By adding up all such component waves, the resultant waves at any time $t = n \cdot \Delta t$ may be determined.

Fig. 61 illustrates the graphical application of this formula to a

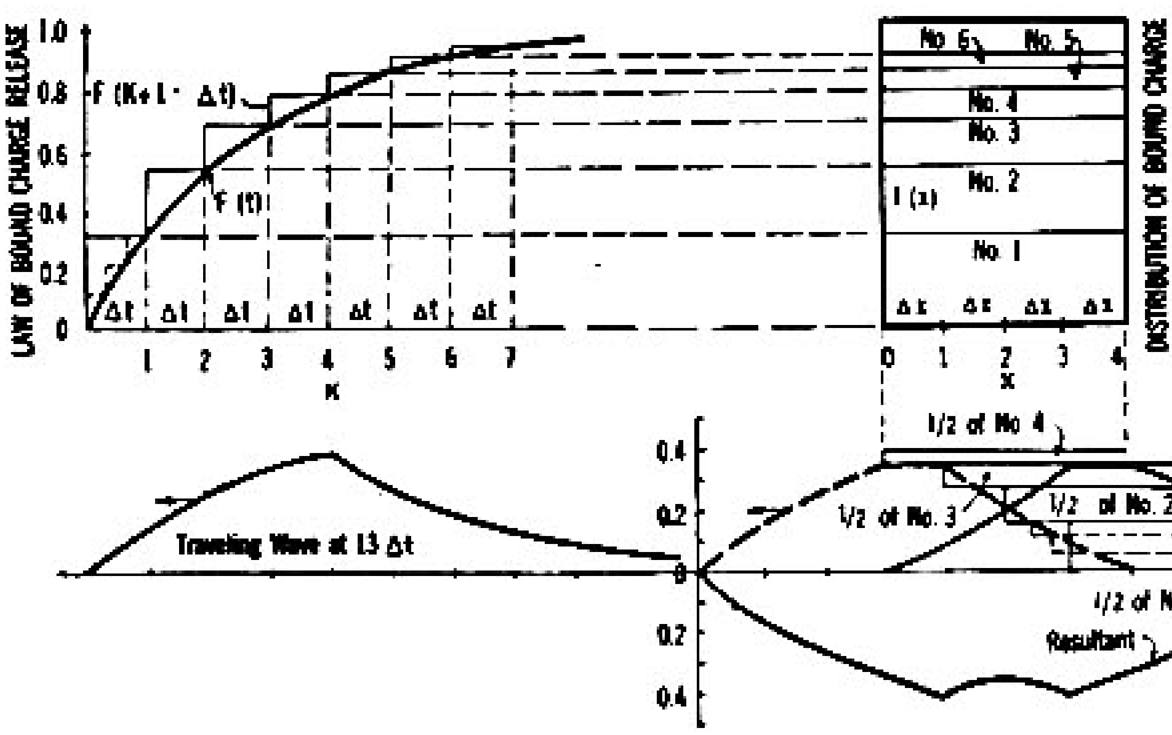


FIG. 61.—Graphical Method for Determining Wave Shapes

rectangular bound charge and a law of cloud discharge $F(t)$. The base of $F(t)$ is divided into equal time increments Δt, and the base of $2f(x)$ into corresponding equal space increments $\Delta x = v \cdot \Delta t$. For ordinary purposes the two increments are equal if t is in microseconds and x in thousands of feet. At the end of $n = 3$ time increments, one half of Block 1 has formed and moved to the right a distance $3 \cdot \Delta x$, and the other half has moved an equal distance to the left. Block 2 has moved to the right and left a distance $2 \cdot \Delta x$, and Block 3 a distance $1 \cdot \Delta t$. Block 4 has just started to form and move out. Only the blocks which have moved to the right are shown in the figure. It is advisable to subdivide the increments Δt for the larger steps, as indicated for Block 1. The resultant of the two traveling waves has been plotted upside down to avoid interference. Of course, for a rectangular bound charge the front of the wave is identical in shape with $F(t)$, and the complete wave is readily obtainable by taking half the ordinates between two $F(t)$ curves displaced a distance apart equal to the length L of the bound charge as shown in Fig. 62.

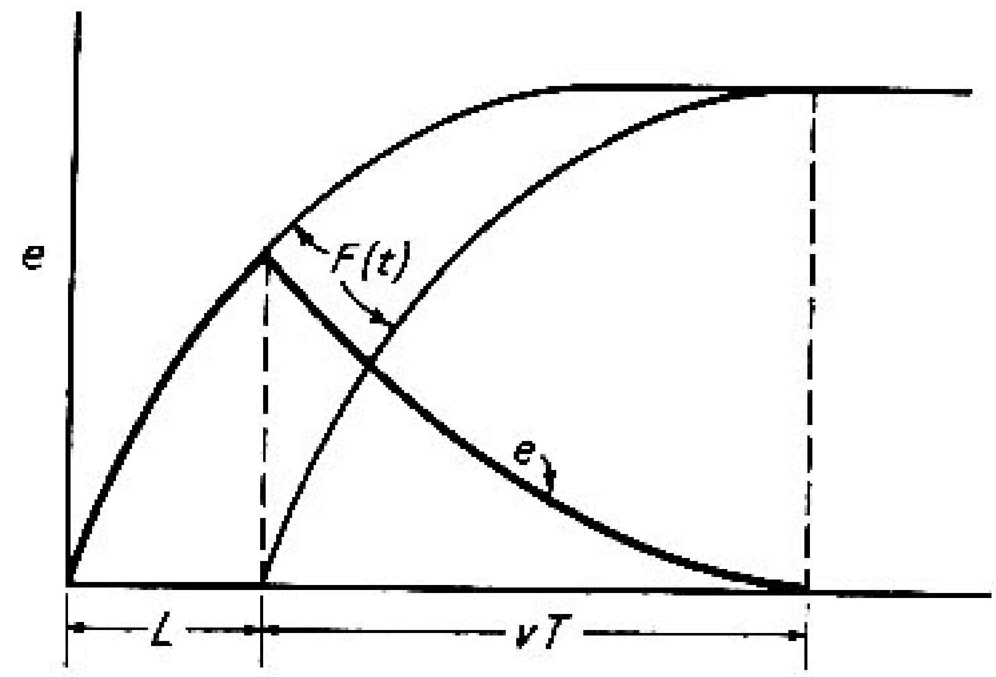
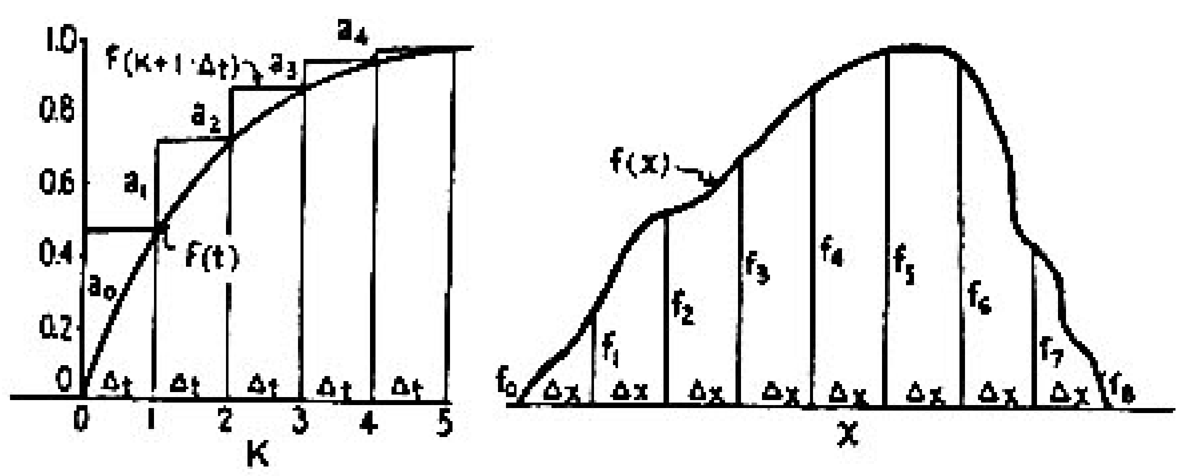


FIG. 62.—Graphical Method

The above graphical method is awkward for other than rectangular bound charges. To study the effect of the bound charge distribution on the shape of the traveling waves it is preferable to use a tabular method for solving (257). The method is shown in Fig. 63. The increments ΔF are given as $(a_0, a_1, a_2, \ldots)$, and the ordinates of $f(x)$ are $(f_0, f_1, f_2, \ldots)$. The table is then filled in as indicated.



K	ΔF_K	$\Delta F_K \cdot f[x-(n-K)\Delta t]$													
											$(n-4)\Delta t$	$(n-3)\Delta t$	$(n-2)\Delta t$	$(n-1)\Delta t$	$n\Delta t$
0	a_0						a_0f_0	a_0f_1	a_0f_2	a_0f_3	a_0f_4	a_0f_5	a_0f_6	a_0f_7	a_0f_8
1	a_1					a_1f_0	a_1f_1	a_1f_2	a_1f_3	a_1f_4	a_1f_5	a_1f_6	a_1f_7	a_1f_8	
2	a_2				a_2f_0	a_2f_1	a_2f_2	a_2f_3	a_2f_4	a_2f_5	a_2f_6	a_2f_7	a_2f_8		
3	a_3			a_3f_0	a_3f_1	a_3f_2	a_3f_3	a_3f_4	a_3f_5	a_3f_6	a_3f_7	a_3f_8			
4	a_4		a_4f_0	a_4f_1	a_4f_2	a_4f_3	a_4f_4	a_4f_5	a_4f_6	a_4f_7	a_4f_8				
.	.					......	etc.	......							
.	.														
$\Sigma =$					$\sum_0^n \Delta F_K \cdot f[x-(n-K)\Delta t]$										

FIG. 63.—Tabular Method for Determining Wave Shapes

The effects of the law of cloud discharge $F(t)$ and the length of rectangular bound charge are shown in Figs. 64 and 65. A comparison of these wave shapes with those of natural lightning lends weight to the assumption that a good approximation for the law of cloud discharge is

$$F(t) \cong (1 - \varepsilon^{-at}) \tag{258}$$

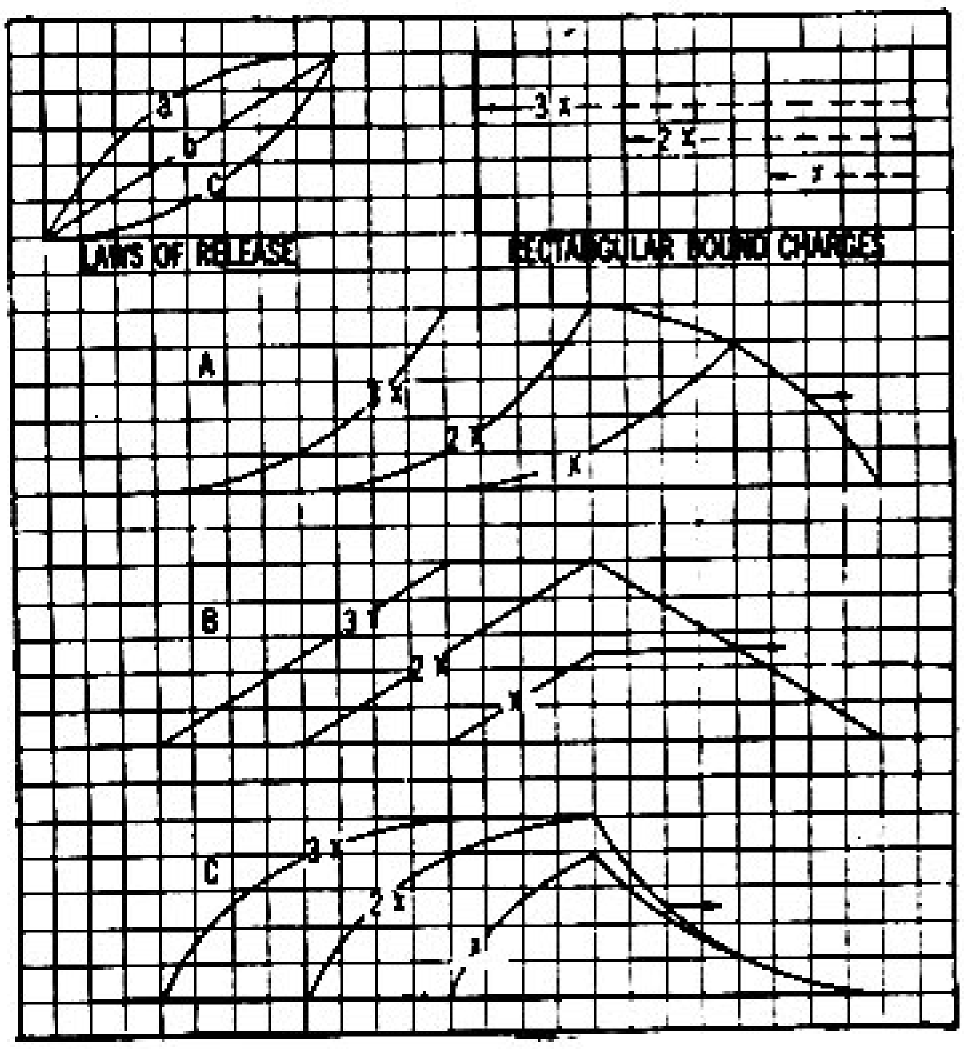


FIG. 64.—Effect of the Law of Discharge on the Shape of Traveling Waves

The shape of the direct stroke waves follows from (250). The sharp corners in these waves are due, of course, to the rectangular bound charge.

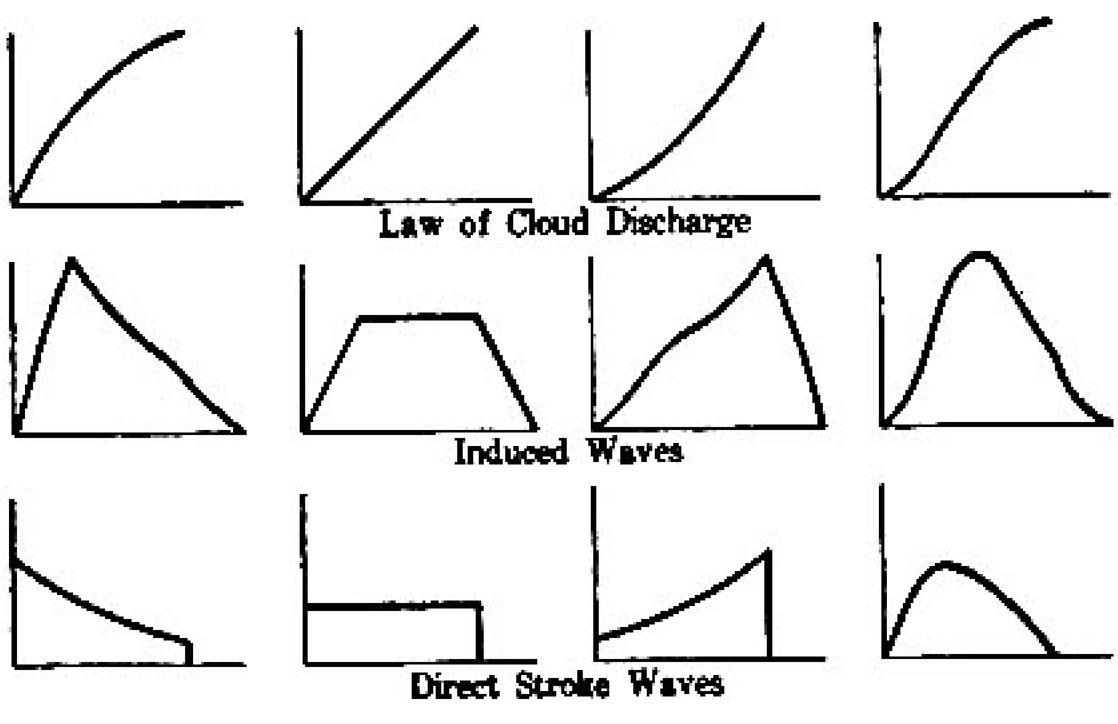


FIG. 65.—Induced and Direct Stroke Waves Corresponding to Different Laws of Cloud Discharge

In Fig. 66 is shown the effect of the time of discharge on the wave shapes from rectangular bound charges. Here it is evident that L must exceed $v\,T$ if the crest of the traveling wave is to reach the value corresponding to instantaneous cloud discharge. The slower the rate of discharge the lower the crest and the longer the tail of the traveling wave. This is a necessary consequence of the fact that all waves originating from a given bound charge must contain exactly the same

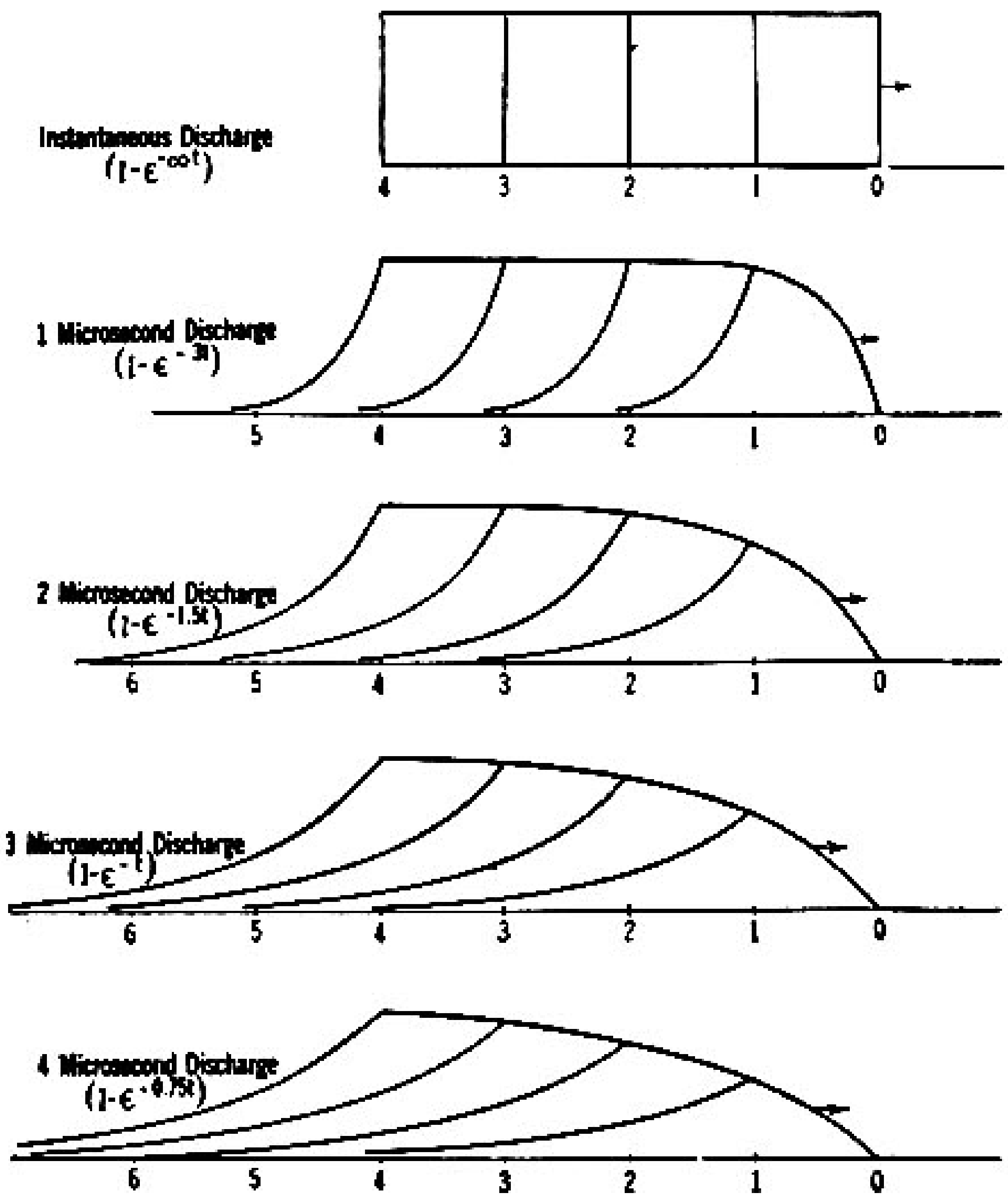


FIG. 66.—Traveling Waves from Rectangular Distribution of Bound Charge

energy (neglecting line losses), regardless of the rate at which that charge is released. The original bound charge represents the storage of a definite amount of energy, and 25 per cent of this energy must

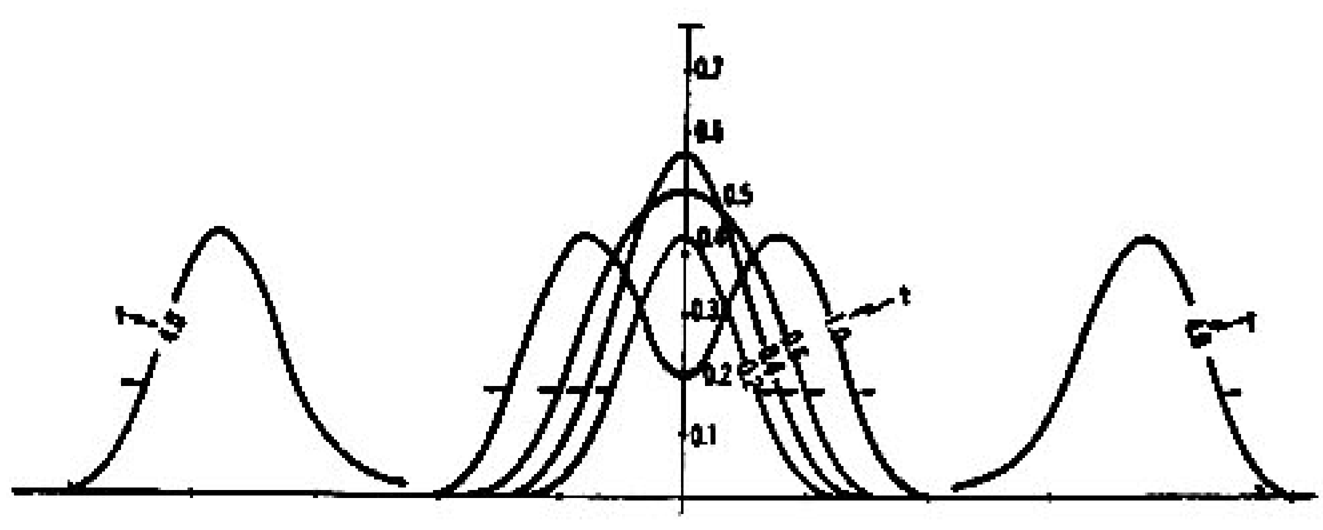


FIG. 67.—Formation of Traveling Waves

appear in each completely developed potential wave, and 25 per cent in each companion current wave. Thus a decrease in crest values must be compensated for by an increase in length.

Fig. 67 illustrates the development of traveling waves during their formation stage, showing how the potential distribution starts from zero, builds up and spreads out until a maximum is reached, then recedes as the two traveling waves separate out and move away in opposite directions.

Fig. 68 shows the traveling waves given by a rectangular bound charge released in zero, one, and three microseconds respectively. One set of curves shows how, as the time of discharge increases, both the maximum crest and the crest of the free traveling waves decrease in value and approach coincidence. The other set of curves shows the maximum potential on the line as a function of time.

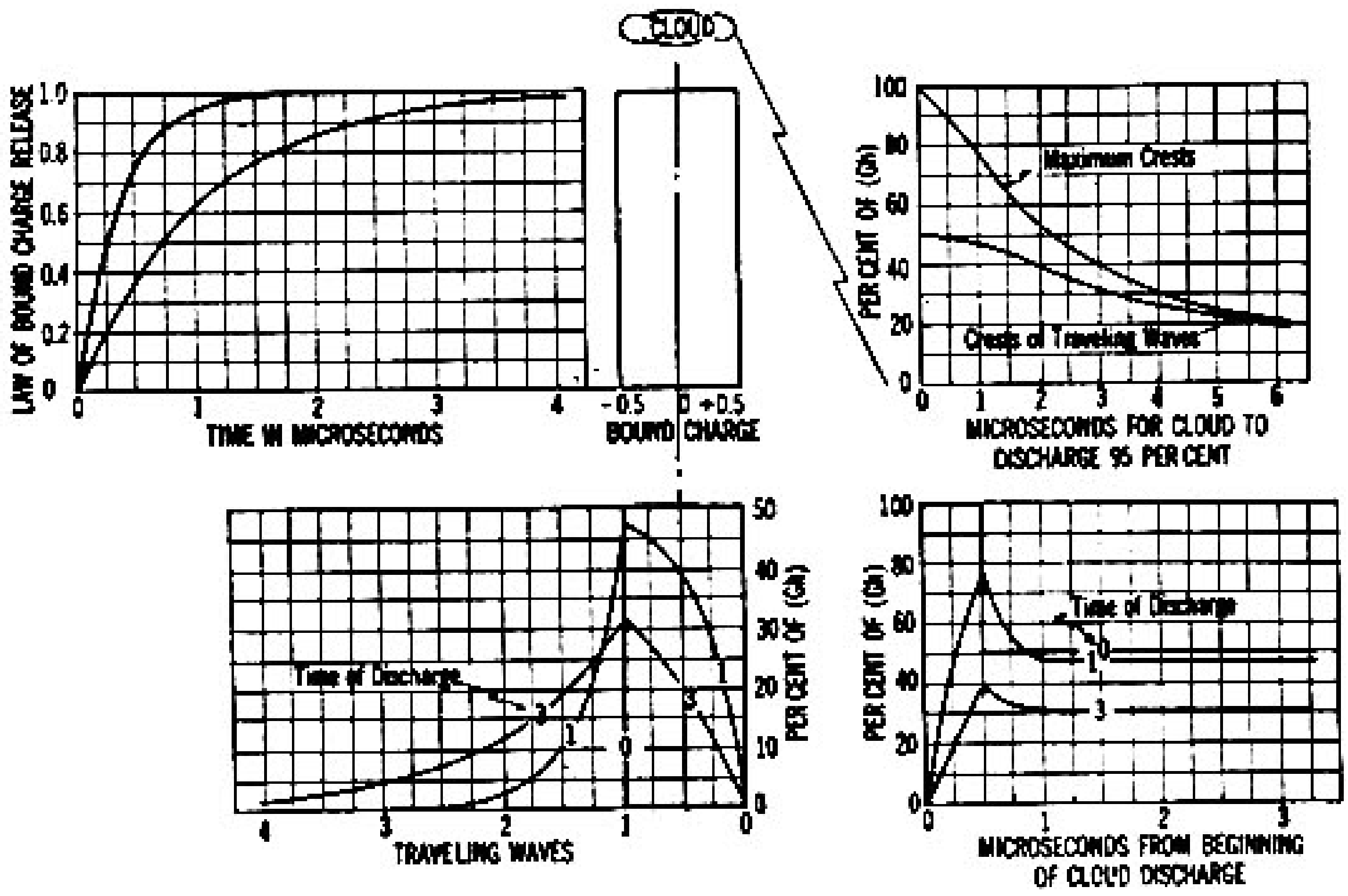


FIG. 68.—Wave Shapes Derived from Rectangular Bound Charge Released at Different Rates

Fig. 69 is the same as Fig. 68 but for a peaked, instead of a rectangular, bound charge.

Fig. 70 illustrates the dominating influence of the law of cloud discharge on the shape of the traveling wave. Rectangular and peaked bound charges having the same total charge and the same amplitude are shown. As the time of cloud discharge increases, the shapes of the traveling waves for both bound charges approach coincidence, excepting for the toe and cap of the wave. This suggests introducing an equivalent rectangular bound charge having a length

$$L = \frac{\int_{-\infty}^{+\infty} f(x) \cdot dx}{f(0)} \tag{259}$$

where $f(x)$ is the distribution of bound charge, and $f(0)$ its crest value. It has been found from numerous calculations of specific cases that

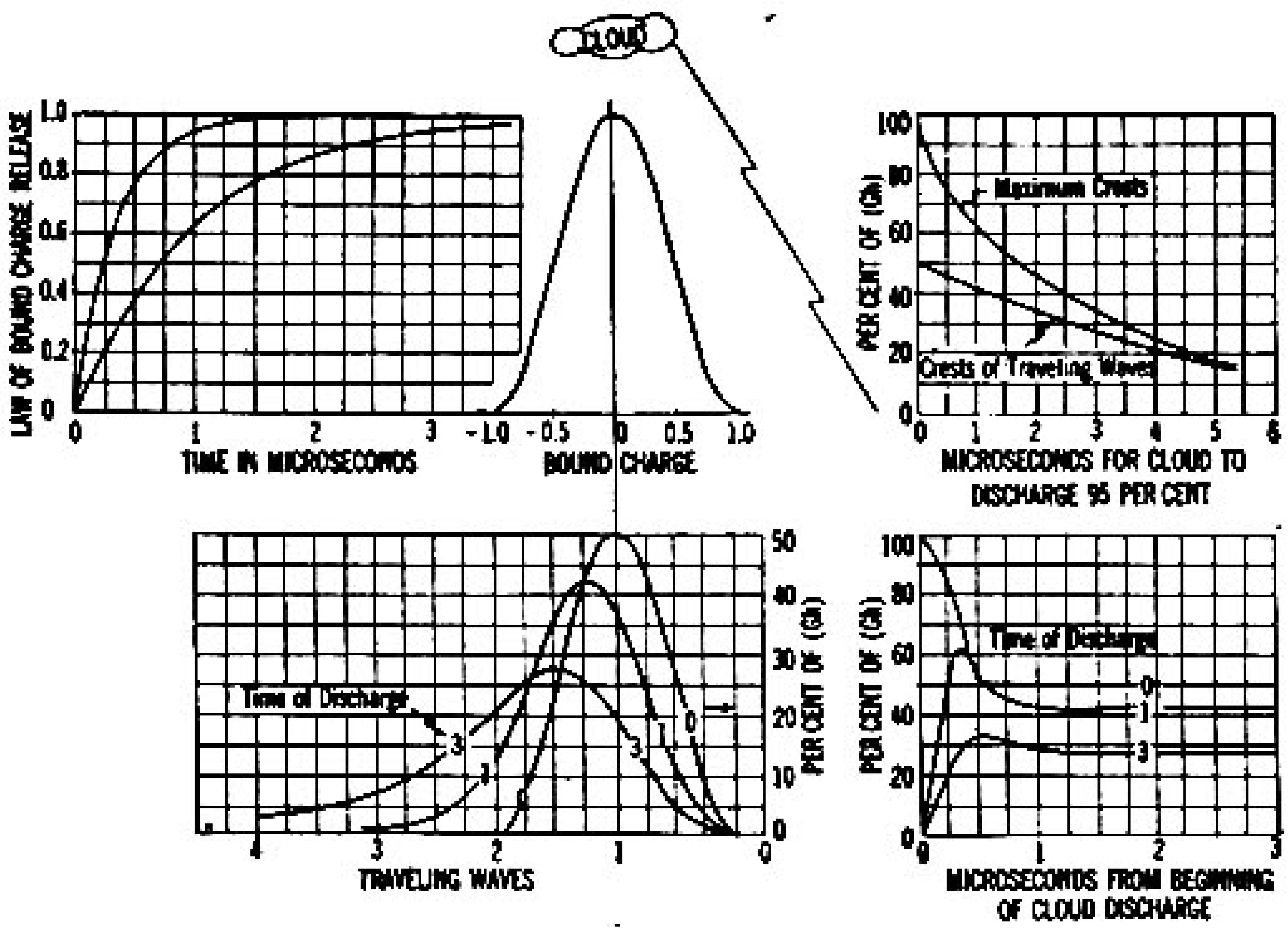


FIG. 69.—Wave Shapes Derived from Peaked Bound Charge Released at Different Rates

this equivalent rectangular bound charge is a good enough approximation for calculating the wave shape, and of course its use greatly

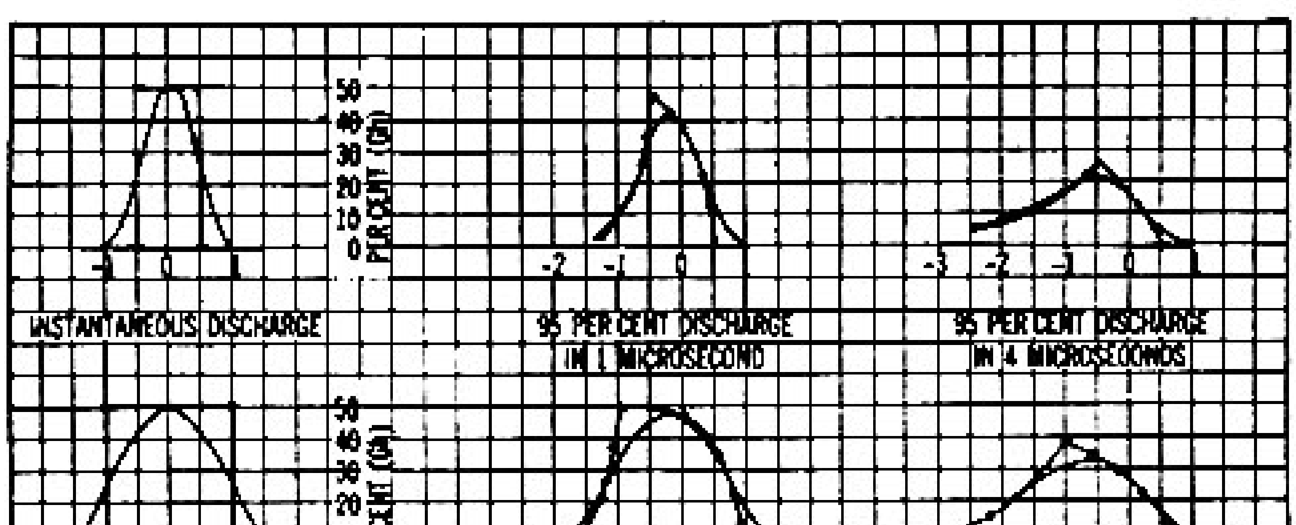


FIG. 70.—Similarity of Wave Shapes from Different Distributions of Bound Charge

simplifies the determination of the wave shape, since the method of Fig. 62 may be used.

If there are no ground wires, the maximum potential on any line wire corresponding to instantaneous cloud discharge is (hG). Taking the finite rate of discharge and the distribution of bound charge into consideration, the voltages are

$$\left.\begin{aligned} e &= \alpha\ Gh = \text{maximum voltage at center of disturbance} \\ e' &= \alpha'\ Gh = \text{maximum crest of traveling wave} \end{aligned}\right\} \qquad (260)$$

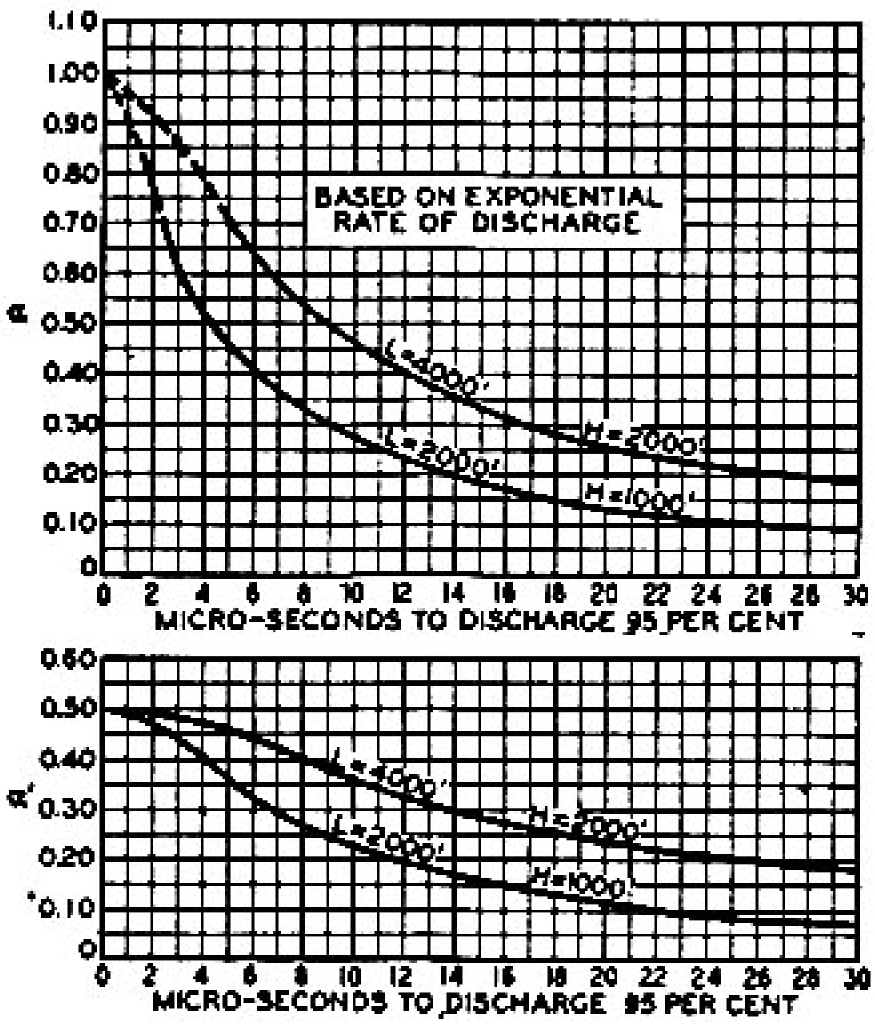


FIG. 71.—α for Induced Voltages (Exponential Discharge)

where α and α' are functions involving $F(t)$ and $f(x)$. These reduction factors can not exceed the limiting values (corresponding to instantaneous discharge) $\alpha = 1.0$ and $\alpha' = 0.5$, respectively. These factors have been plotted in Fig. 71 for rectangular bound charges of 2000 and 4000 ft. length, and for $F(t) = (1 - \varepsilon^{-at})$.

If $F(T) = 0.05$, then $a = 3\ T$, and the crest of the traveling wave, reached at $t = L\ v$, Fig. 62, is

$$e' = \frac{1}{2}\,(1 - \varepsilon^{-(3L/vT)})\ G\ h \qquad (261)$$

At $x = 0$ (midpoint of the bound charge distribution) the tail of each wave is

$$e' = 0.5\,[1 - \epsilon^{-a(t - L/2v)}]\ G\ h \quad \text{for } t \leq L\ v$$

This is a maximum at $t = L\ v$. The maximum voltage for both waves then is

$$e = [1 - \epsilon^{-(1.5L/vT)}]\ G\ h \qquad (262)$$

Comparing (260), (261), and (262) it is evident that

$$\alpha' = 0.5\ (1 - \epsilon^{-(3L/vT)}) \qquad (263)$$

$$\alpha = 1.0\ (1 - \epsilon^{-(1.5L/vT)}) \qquad (264)$$

If $T = 0$, instantaneous discharge, $\alpha' = 0.5$ and $\alpha = 1.0$.

From the foregoing analysis a number of interesting facts are obvious: *

1. The total length of the free traveling wave in thousands of feet is the length of the bound charge plus the time of cloud discharge in microseconds: $L + v\ T$.

* Discussion by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 49, p. 929.

2. The length of the front of the free traveling wave is the smaller of L and $v\,T$.
3. The crest of the wave is lower, the longer T and the shorter L.
4. The shape of the traveling wave is dominated by $F(t)$, and the influence of $f(x)$ is quite subsidiary thereto. Therefore, an equivalent rectangular bound charge may be substituted for the actual distribution, thereby greatly simplifying the analysis.
5. The greater T, the more nearly does the maximum resultant voltage at the center of disturbance equal the crest of the traveling wave.

It is apparent, then, that a cathode-ray oscillogram of an induced surge is a fairly complete record of its entire history, including the law of cloud discharge and the bound charge from which it originated.

For a few simple cases, Equation (257) may be solved analytically. As an example, suppose that the cloud is a uniformly charged sphere whose center is at a height H above the ground plane and its image $(-Q)$ at a depth $(-H)$ below the ground plane. Now it is shown in texts on electrostatics that the external field of a spherical volume charge is the same as that of a point charge having the same charge Q and situated at the center of the sphere. Therefore the vertical component of gradient near the surface of the ground of the equivalent point charge and its image is

$$G(x) = \frac{2\,Q}{(H^2 + x^2)} \cos\theta = \frac{2\,QH}{(H^2 + x^2)^{3/2}}$$

This is a maximum at $x = 0$ and is

$$G_{\max} = \frac{2\,Q}{H^2}$$

The equivalent rectangular bound charge has a length

$$L = \frac{2\,QH}{(2\,Q/H^2)}\int_{-\infty}^{+\infty} \frac{dx}{(H^2 + x^2)^{3/2}} = 2\,H$$

The cloud potential at a radius R is approximately

$$V \cong \frac{Q}{R} \cong \frac{H^2}{2R}\,G_{\max}$$

The distribution of potential due to the instantaneous release of the bound charge is

$$2\,f(x) = h\,G(x) = \frac{2\,Q\,h\,H}{(H^2 + x^2)^{3/2}}$$

Assume that the discharge is linear

$$F(t) = t/T \qquad \text{for } t \leq T$$
$$= 1 \qquad \text{for } t \geq T$$

Each traveling wave then is

$$e' = \int_0^t f\,[x \pm v\,(t - \tau)]\,\frac{\partial\,F(\tau)}{\partial\,\tau}\,d\tau$$

$$= \int_0^t \frac{Q\,h\,H}{T}\,\frac{d\tau}{[H^2 + (x \pm vt)^2 \mp 2\,v\,(x \pm vt)\,\tau + v^2\,\tau^2]^{3/2}}$$

$$= \frac{h\,Q}{v\,T\,H}\left[\frac{(x \pm vt)}{\sqrt{H^2 + (x \pm vt)^2}} - \frac{x}{\sqrt{H^2 + x^2}}\right] \text{for } t \leq T$$

$$= \frac{h\,Q}{v\,T\,H}\left[\frac{(x \pm vt)}{\sqrt{H^2 + (x \pm vt)^2}} - \frac{(x \pm vt \mp vT)}{\sqrt{H^2 + (x \pm vt \mp vT)^2}}\right] \text{for } t \geq T$$

Thus up to $t = T$ the potential consists of a pure traveling wave and a superimposed stationary distribution. After $t = T$ the wave is fully developed and the stationary distribution has disappeared. The traveling wave has a maximum at $t = T$ and $x = v\,T/2$ of

$$E' = \frac{2\,h\,Q}{H\,v\,T}\,\frac{1}{\sqrt{1 + (2\,H/vT)^2}} = \frac{2\,h}{v\,H}\,\frac{I}{\sqrt{1 + (2\,H/vT)^2}}$$

where $I = Q/T$ is the discharge current.

The reduction factor is

$$\alpha' = \frac{E'}{G\,h} = \frac{H}{vT}\,\frac{1}{\sqrt{1 + (2\,H/vT)^2}} = \frac{L}{2\,vT}\,\frac{1}{\sqrt{1 + (L/vT)^2}}$$

The resultant potential due to the combination of both the forward and backward waves is their sum

$$e = \frac{h\,Q}{H\,vT}\left\{\frac{(x + vt)}{\sqrt{H^2 + (x + vT)^2}} + \frac{(x - vt)}{\sqrt{H^2 + (x - vT)^2}} - \frac{2\,x}{\sqrt{H^2 + x^2}}\right\} \text{for } t \leq T$$

$$= \frac{h\,Q}{H\,vT}\left\{\frac{(x+vt)}{\sqrt{H^2+(x+vT)^2}}+\frac{(x-vt)}{\sqrt{H^2+(x-vT)^2}}\right.$$
$$\left.-\frac{x+v\,(t-T)}{\sqrt{H^2+(x+vt-vT)^2}}-\frac{x-v\,(t-T)}{\sqrt{H^2+(x-vt+vT)^2}}\right\}\text{ for } t \geq T$$

This potential reaches a maximum at $x = 0$ and $t = T$ of

$$E = \frac{2\,h}{v\,H}\,\frac{I}{\sqrt{1+(H/vT)^2}}$$

The reduction factor therefore is

$$\alpha = \frac{E}{G\,h} = \frac{H}{vT}\,\frac{1}{\sqrt{1+(H/vT)^2}} = \frac{L}{2\,vT}\,\frac{1}{\sqrt{1+(L/2\,vT)^2}}$$

If, as probably is the case, vT is large compared to L, the above equations reduce (in practical units) to:

$$E' \cong E \cong 60\,\frac{h}{H}\,I = \alpha\,G\,h \text{ volts}$$

$$\alpha' \cong \alpha \cong \frac{H}{1000\,T} = \frac{0.06\,I}{G\,H}$$

$$G \cong \frac{60\,I\,T}{H^2}$$

where I is in *amperes*, h and H in *feet*, T in *microseconds*, and G in *kilovolts per foot*. These equations have been plotted * in Fig. 72 for $G = 100$.

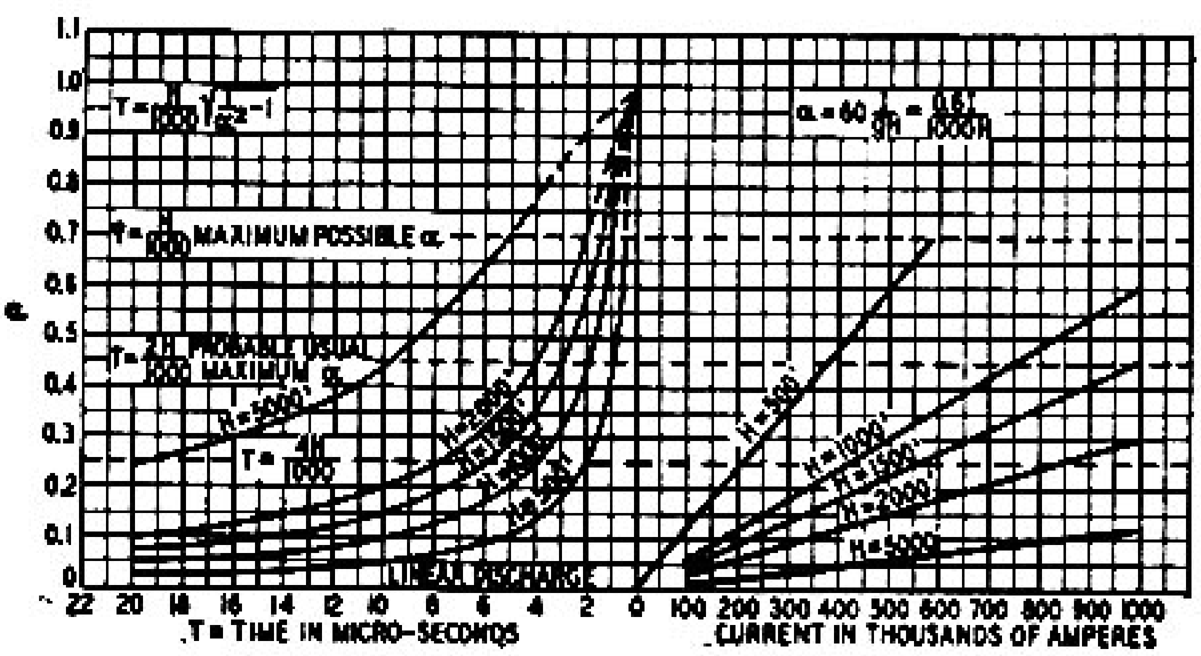


FIG. 72.—Variation of α with Lightning Currents and Time of Cloud Discharge for Different Cloud Heights

(Based upon point cloud and linear rate of cloud discharge)

* "Lightning," by F. W. Peek, Jr., *A.I.E.E. Trans.*, 1931.

LINEAR DISCHARGE

Line 50 Ft. High (No Ground Wires)

Cloud Height Feet	Lightning Current Amperes (Assumed)	α (Calculated)	Induced Voltages Kv. (Calc.)	Approximate Time of Discharge μ Sec. (Calc.)
500	600,000	0.70	3600	0.5
1000	600,000	0.36	1800	3.
2000	600,000	0.18	900	11.
500	200,000	0.24	1200	2.
1000	300,000	0.18	900	6.
2000	300,000	0.09	450	22.
5000	200,000	0.024	120	200.
Line 80 Ft. High (No Ground Wires)				
500	600,000	0.70	5760	0.5
1000	600,000	0.36	2880	3.
2000	600,000	0.18	1440	11.
1000	300,000	0.18	1440	6.
5000	200,000	0.024	190	200.

EXPONENTIAL DISCHARGE

Line 50 Ft. High

500	200,000	0.25	1250	6.
1000	410,000	0.25	1250	11.
2000	840,000	0.25	1250	21.
Line 80 Ft. High				
500	200,000	0.25	2000	6.
1000	410,000	0.25	2000	11.
2000	840,000	0.25	2000	21.
500	90,000	0.125	1000	12.
1000	195,000	0.125	1000	22.
2000	400,000	0.125	1000	45.

To estimate the surge voltage induced by lightning it is not permissible to assume arbitrary values for any of the parameters in the above equations which violate the known confines of any of the others. The present indications are that the limits of the different parameters are as follows:

G = 100 kv. per ft. (which value is generally agreed upon).

H is from 500 to 5000 ft. (based on meteorological survey).

I, except for unusually heavy stroke, very likely does not exceed 300,000 amperes (based on field measurements of currents in towers struck by lightning).

$T > 2\,H/1000$, since the velocity of discharge certainly can not exceed the velocity of light. (Examination of numerous cathode-ray oscillograms of natural lightning fails to show a discharge faster than 10 ms., some last for over 100 ms., and the average is about 25 to 30 ms.)

The attached table * indicates the range covered by different assumptions. It is thus evident that, although induced voltages may not be dangerous for the more highly insulated lines, they become of increasing importance as line insulation is decreased.

SUMMARY OF CHAPTER IX

Traveling waves due to lightning may be caused either by electrostatic induction or by a direct stroke. The voltage of an induced surge depends upon the time of cloud discharge, the initial electrostatic field gradient of the cloud, and the distribution of the bound charge. These factors in turn are tied in with the maximum current in the lightning stroke, and the potential, height, and length of the cloud. Ground wires practically halve the magnitude of induced surges. On the assumption that the current in a lightning stroke does not exceed 300,000 amperes and that the time of cloud discharge is at least 10 ms., it is doubtful if an induced surge is ever as high as 1000 kv., and more probably 500 kv. is the upper limit. On this basis, lines of 66 kv. or more are immune from trouble as far as induced surges are concerned. On the other hand, a direct lightning stroke may reach voltages of the order of 10,000 kv. The typical lightning waves, whether due to electrostatic induction or direct stroke, are of the same general shapes. Specifically, the direct stroke is given by

$$E = Z\,I = Z\,Q_0\,\frac{\partial\,F(t)}{\partial\,t}$$

and the induced voltage by

$$e = \int_0^t f\,[x \pm v\,(t - \tau)]\cdot\frac{\partial\,F(\tau)}{\partial\,\tau}\,d\tau$$

where $F(t)$ is the law of cloud discharge and $f(x)$ the distribution of bound charge. The integral can not be evaluated explicitly, except in a few simple cases, but numeri-

* "Lightning," by F. W. Peek, Jr., *A.I.E.E. Trans.*, 1931.

cal results are easily obtained by means of graphical and tabular methods. From these solutions it is found that the magnitude of the induced voltage is

$$e = \alpha G h ,$$

where α is a function depending on $f(x)$ and $F(t)$, and for all practical purposes

$$F(t) \cong (1 - \varepsilon^{-at})$$

The function α is plotted in Fig. 71. The salient characteristics of induced surges are:

(1) *Total Length of Waves* = *Length of Bound Charge* + *Time of Cloud Discharge.*

(2) *Front of Traveling Wave* = *Length of Bound Charge.*

(3) *Crest of the Wave* decreases as *Time of Discharge* increases or as *Current in Stroke* decreases.

(4) *Shape of the Wave* is principally dependent upon the *Law of Cloud Discharge.*

(5) *The Cathode-Ray Oscillogram of an Induced Surge* is a practically *Complete Record of the History of that Surge.*

CHAPTER X

GROUND WIRES *

Ground wires were originally used on transmission lines as a protection against induced lightning waves. In that capacity they practically halve the magnitude of the impulse, but they function the more efficiently, the closer they are to the power conductors. Within the last few years, however, there has been growing reliable evidence, of both a theoretical and experimental character, that most of the outages due to lightning which occur on high-voltage lines are caused by direct strokes rather than induced voltages; and this has led to a somewhat different method for employing ground wires, so as to be in a better position to intercept the direct stroke. Ground wires also exercise a number of subsidiary effects, among which may be mentioned their effect on:

1. Zero sequence reactance of the transmission line.
2. Telephone interference.
3. Corona.
4. Attenuation of traveling waves.
5. Reduction in surge impedance.
6. Relaying possibilities.

Induced Surges with Ideal Ground Wires.—An ideal ground wire is one which is perfectly grounded at all points throughout its length, and thus is always at zero potential. It therefore differs from an actual ground wire grounded at definite intervals through finite tower footing resistances, in that it is free from successive reflections. A comparison of the results of this section with those of the next section shows that the traveling waves a few towers removed from the initial distribution of bound charge are practically the same in either case; but it is much easier to compute the free wave on the basis of ideal ground wires.

Consider an overhead system having m ideal ground wires and $(n - m)$ line wires. Number the ground wires from 1 to m inclusive

* "Critique of Ground Wire Theory," by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 49.

and the line wires from $(m + 1)$ to n inclusive. Let the initial cloud field gradient be G and the corresponding bound charge distributions be $f(x)$. Then under equilibrium conditions just prior to cloud discharge the bound charges on the conductors are given by Equations (252), from which the initial bound charges $(Q_1 \ldots Q_n)$ may be determined. If the cloud discharges instantaneously, Equations (253) apply, and are sufficient to determine the n unknowns $(Q_1' \ldots Q_m' \; V_{m+1} \ldots V_n)$. According to (254), these released bound charges move out as pairs of exactly similar traveling waves moving in opposite directions, so that by (253) and (256) the free traveling wave f_k on conductor k is given by

$$V_k = 2 f_k = p_{k1} Q_1' + \ldots + p_{km} Q_m' + p_{k(m+1)} Q_{(m+1)} + \ldots + p_{kn} Q_n \quad (265)$$

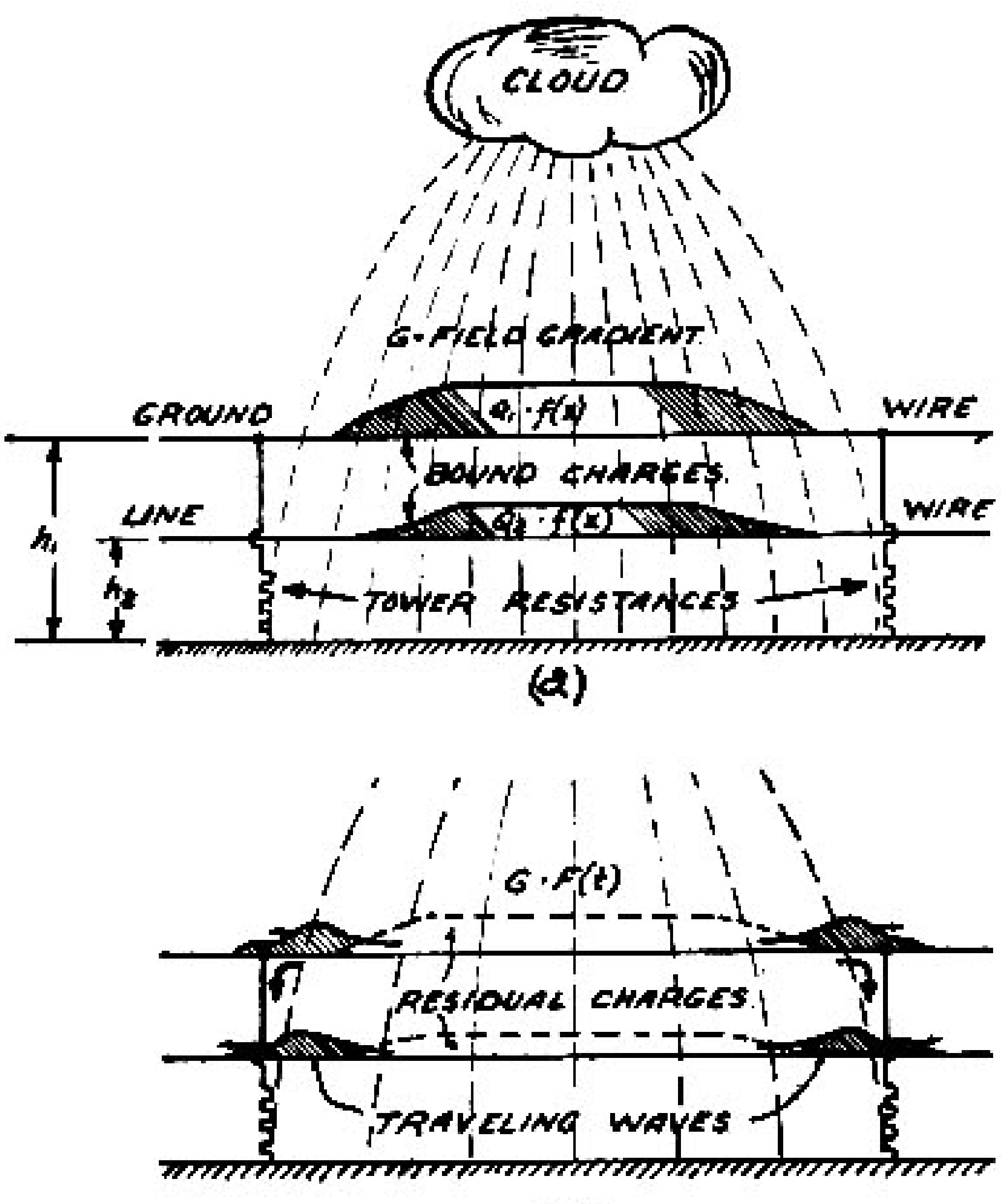


FIG. 73.—Release of Bound Charges on Line and Ground Wires

(a) Before cloud discharge
(b) During cloud discharge

Now suppose that the cloud discharges so that at any instant t the gradient is given by $G[1 - F(t)]$, where $F(t)$ is the law of cloud discharge, assumed to be uniform over the bound charge distribution $2f(x)$. The bound charges will be released proportionally to the decrease in the cloud gradient, and the residual charge on any line conductor at any instant is $Q_k[1 - F(t)]$. The released portions of the bound charges become pairs of traveling waves. The shapes of the traveling waves on all line conductors depend only on $f(x)$ and $F(t)$ and are therefore similar, as given by (257). For convenience, let the forward and reverse traveling waves be represented by $Q \cdot \phi(x, t)$ and $Q \cdot \psi(x, t)$ respectively, where the functions $\phi(x, t)$ and $\psi(x, t)$ are to be determined from (257). The total potential at x and t therefore is, calling $\beta(x, t) = [1 - F(t)]\, 2 f(x)$

$$\left.\begin{aligned} E_k = h_k G\cdot\beta(x, t) + [p_{k1} Q_1 + \ldots + p_{kn} Q_n]\,\beta(x,t) + [p_{k1} Q_1' \\ + \ldots + p_{km} Q_m' + p_{k(m+1)} Q_{(m+1)} + \ldots + p_{kn} Q_n] \\ [\phi(x, t) + \psi(x, t)] \end{aligned}\right\} \quad (266)$$

The first term on the right is the potential due to the residual field of the cloud; the second term is that due to the residual charges on the line and ground conductors; the third term that due to the traveling-wave components. But by (252) the sum of the first two terms on the right is zero, and the n equations of type (266) are then identical with those of (253) multiplied by $[\phi(x, t) + \psi(x, t)] = a$. By (252)

$$\begin{aligned} p_{k(m+1)} Q_{(m+1)} + \ldots + p_{kn} Q_n \\ = -(G h_k + p_{k1} Q_1 + \ldots + p_{km} Q_m) \end{aligned} \quad (267)$$

Hereby (266) reduces to

$$\begin{aligned} - E_k + p_{k1}(Q_1' - Q_1)\,a + p_{k2}(Q_2' - Q_2)\,a \\ + \ldots + p_{km}(Q_m' - Q_m)\,a = - G h_k a \end{aligned} \quad (268)$$

in which E_k is zero for $k = 1$ to $k = m$ inclusive.

The symbolic determinate then is

$$\begin{array}{cccc|c} a(Q_1'-Q_1) & \ldots a(Q_m'-Q_m) & E_{(m+1)} & \ldots E_n & \\ \hline p_{11} & \ldots p_{1m} & 0 & \ldots 0 & = - a G h_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m1} & \ldots p_{mm} & 0 & \ldots 0 & = - a G h_m \\ p_{(m+1)1} & \ldots p_{(m+1)m} & -1 & \ldots 0 & = - a G h_{(m+1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n1} & \ldots p_{nm} & 0 & \ldots -1 & = - a G h_n \end{array} \quad (269)$$

Solving for $E_{(m+k)}$, there results

$$E_{(m+k)} = \left|\frac{\begin{matrix} p_{11} & \ldots p_{m1} & h_1 \\ \cdot & \cdot & \cdot \\ p_{1m} & \ldots p_{mm} & h_m \\ p_{1(m+k)} & \ldots p_{m(m+k)} & h_{(m+k)} \end{matrix}}{\begin{matrix} p_{11} & \ldots p_{m1} \\ \cdot & \cdot \\ p_{1m} & \ldots p_{mm} \end{matrix}}\right| a G = D a G \quad (270)$$

As an example, consider the previous numerical case in which (since ρ_1 is now to be solved for)

$$
\begin{aligned}
p_{11} &= p_{22} = 2 \log\left(\frac{1200}{\rho_1}\right) \\
p_{12} &= 4.62 \\
p_{13} &= p_{14} = p_{24} = p_{25} = 4.17 \\
p_{15} &= p_{23} = 3.25 \\
p_{33} &= p_{44} = p_{55} = 15.12 \\
p_{34} &= p_{45} = 4.17 \\
p_{35} &= 2.83
\end{aligned}
$$

Since from symmetry $Q_1 = Q_2$ and $Q_3 = Q_5$, the determinate for Q_1 becomes

$$
Q_1 = -G \left| \frac{\begin{matrix} h_1 & p_{13} & (p_{13}+p_{15}) \\ h_3 & p_{34} & (p_{33}+p_{35}) \\ h_3 & p_{33} & 2\,p_{34} \end{matrix}}{\begin{matrix} (p_{11}+p_{12}) & p_{13} & (p_{13}+p_{15}) \\ (p_{13}+p_{15}) & p_{34} & (p_{33}+p_{35}) \\ 2\,p_{13} & p_{33} & 2\,p_{34} \end{matrix}} \right| = -G \left| \frac{\begin{matrix} 50 & 4.17 & 7.42 \\ 40 & 4.17 & 17.95 \\ 40 & 15.12 & 8.34 \end{matrix}}{\begin{matrix} (p_{11}+4.62) & 4.17 & 7.42 \\ 7.42 & 4.17 & 17.95 \\ 8.34 & 15.12 & 8.34 \end{matrix}} \right|
$$

$$
= -\frac{7000\,G}{236.7\,p_{11} + 150} = G\cdot f(\rho_1)
$$

Then by (276), putting $G = 100$ kv. per ft.,

$$
100 = 37.9\left[1 + \frac{0.189}{\sqrt{\rho_1}}\right]\left[\log\frac{1200}{\rho_1} + 0.317\right]\frac{\rho_1}{14.8}
$$

This equation is satisfied by $\rho_1 = 6.6$ and $p_{11} = 10.4$. Therefore the protective ratio for outside conductors is

$$
(\mathrm{P.R.})_5 = \left| \frac{\begin{matrix} 10.4 & 4.62 & 50 \\ 4.62 & 10.4 & 50 \\ 4.17 & 3.25 & 40 \end{matrix}}{\begin{matrix} 10.4 & 4.62 \\ 4.62 & 10.4 \end{matrix}} \right| \frac{1}{40} = 0.384
$$

Representative values for 220-kv. lines based on a cloud gradient of 100 kv. per ft. and instantaneous discharge are shown in Fig. 74; and in the following table there is a comparison between calculations by Hunter's method and results obtained by Peek from tests on model transmission lines. The agreement is quite good. Nevertheless, Hunter's method ignores the fact that the charges on the ground wires are not constant, and are replaced by new charges during the cloud discharge, so that the effective diameter of the corona envelope may also be changing (unless the initial envelope persists throughout the transient).

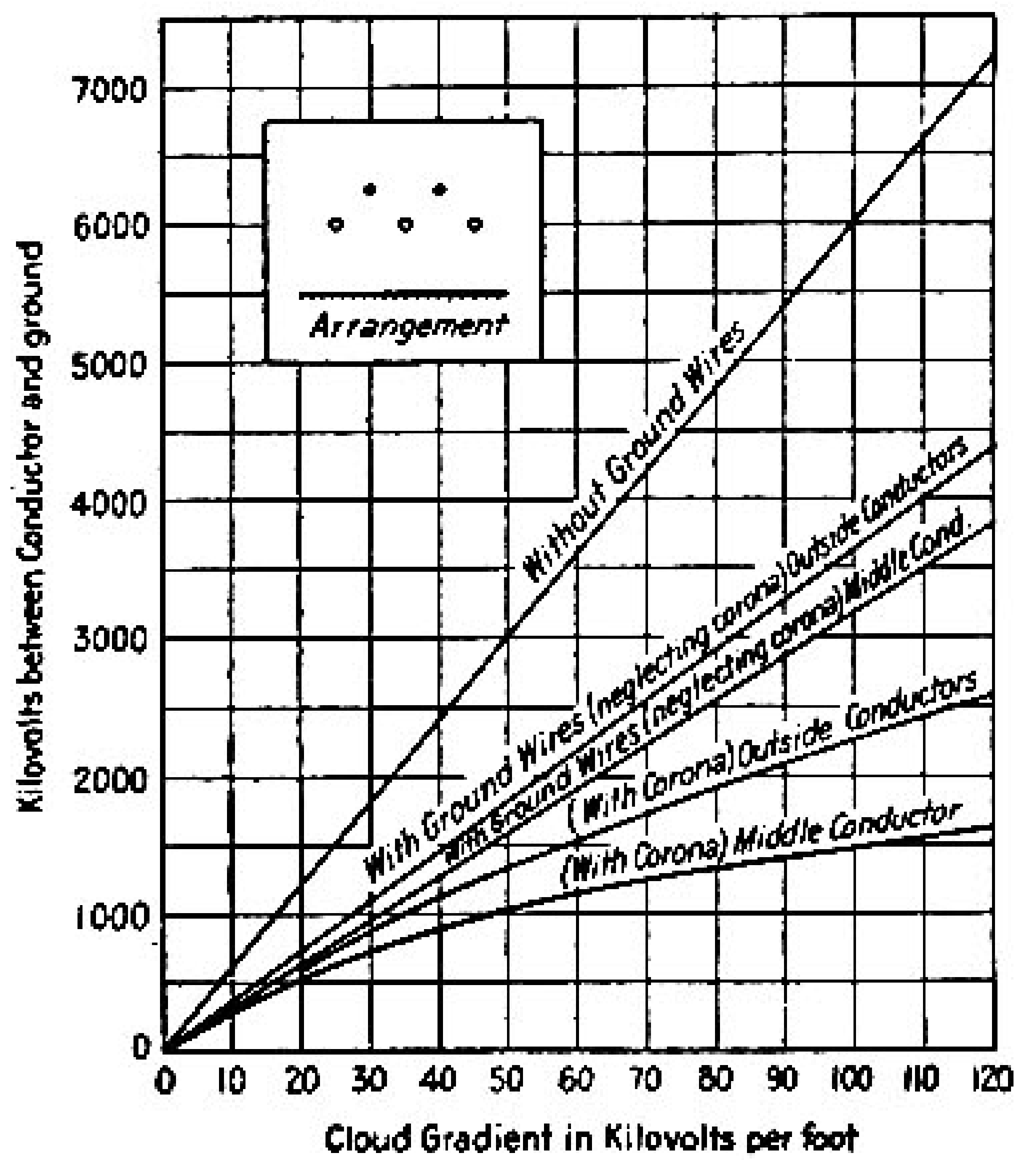


FIG. 74.—Hunter's Correction for Corona

COMPARISON OF CALCULATIONS AND TESTS

Arrangement		Peek's Tests on Models	Hunter's Method
0 0 0 1 2 3	(1) (2) (3)	0.52 0.44 0.52	0.57 0.47 0.57
0 0 0 1 2 3	(1) (2) (3)	0.40 0.34 0.40	0.38 0.25 0.38
0 0 0 1 2 3	(1) (2) (3)	0.34 0.28 0.34	0.31 0.24 0.31
0 0 1 0 0 2 0 0 3	(1) (2) (3)	0.42 0.49 0.56	0.46 0.52 0.56
0 0 1 0 0 2 0 0 3	(1) (2) (3)	0.33 0.38 0.44	0.27 0.33 0.40

Tests on transmission lines* with artificial lightning surges give protective ratios which check very well with the conventional calculations, Equation (272). Equal traveling waves were impressed on all three conductors of a three-phase line. One of these conductors was grounded at several towers 5.5 miles from the impulse generator. When the traveling waves reached the grounding point the potential of that conductor became zero, thus simulating the action of a ground wire during a cloud discharge (the initial charge on the ground wire escapes to

* "Traveling Waves on Transmission Lines with Artificial Lightning Surges," by K. B. McEachron, J. G. Hemstreet, and W. J. Rudge, *A.I.E.E. Trans.*, Vol. 49.

ground and is replaced by a new charge sufficient to maintain the ground wire at zero potential). In the case under consideration, the calculated surge impedances and conductor heights were

$$Z_{11} = 537 \qquad Z_{12} = 107 \qquad h_1 = 45'$$

$$Z_{22} = 532 \qquad Z_{23} = 145 \qquad h_2 = 41'$$

$$Z_{33} = 542 \qquad Z_{31} = 117 \qquad h_3 = 49'$$

All conductors were 0.14-in. radius. The calculated protective ratios for No. 2 and No. 3 conductors are

$$(\text{P.R.})_2 = 1 - \frac{Z_{12}\, h_1}{Z_{11}\, h_2} = 0.78$$

$$(\text{P.R.})_3 = 1 - \frac{Z_{13}\, h_1}{Z_{11}\, h_3} = 0.80$$

From traveling-wave calculations, Equations (285) and (287), the transmitted voltages are

$$e_1'' = \frac{2\,R}{2\,R + Z_{11}}\, e_1$$

$$e_2'' = e_2 - \frac{Z_{12}}{2\,R + Z_{11}}\, e_1$$

$$e_3'' = e_3 - \frac{Z_{13}}{2\,R + Z_{11}}\, e_1$$

Conductor 1 was grounded through different resistances R (at one, two, and three towers respectively), and the transmitted voltages measured. A comparison between the test and calculated values is given in the following table:

R	Measured on Conductors			Calculated Waves			Calculated Ratio e''/e	
	No. 1	No. 2	No. 3	No. 1	No. 2	No. 3	No. 2	No. 3
∞	168	177	178	....	....	....	1.00	1.00
76.4	40	146	....	37	151	148	0.850	0.832
20.7	34	145	....	12	146	144	0.825	0.810
5.5	4.3	142	143	3.4	144	142	0.813	0.798
0	....	....	....	0	143	141	0.807	0.792

It will be observed that the calculated and test values check very well, and that the ratio of the transmitted and incident waves checks the calculated (P.R.). It is also evident that the tower footing resistance does not have much effect on the protective ratio.

The general case of m ground wires tied together at the tower and grounded through a resistance R presents a rather awkward situation to solve. It may be shown, however, that the reflections depend only on the waves on the ground wires. The transition-point conditions are:

$$\left.\begin{aligned}
&e_k + e_k' = e_k'' \text{ on all conductors} \\
&e_1'' = e_2'' = \ldots = e_m'' = R\,I \\
&(i_1 + i_1') + \ldots + (i_m + i_m') = i_1'' + \ldots + i_m'' + I \\
&i_{(m+1)} + i_{(m+1)}' = i_{(m+1)}'' \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&i_n + i_n' = i_n''
\end{aligned}\right\} \tag{277}$$

Therefore

$$\left.\begin{aligned}
&e_k - e_k' - e_k'' = e_k - e_k' - (e_k + e_k') = -2\,e_k' \\
&e_r' = e_r'' - e_r = (e_1 + e_1') - e_r \quad \text{if} \quad r \leq m
\end{aligned}\right\} \tag{278}$$

Making these substitutions and rearranging, there finally result $(n - m + 1)$ simultaneous equations relating $(e_1', e_{m+1}' \ldots e_n')$

$$\left.\begin{aligned}
&(Y_{(m+1)1} + \ldots + Y_{(m+1)m})\,e_1' + Y_{(m+1)\,(m+1)}\,e_{(m+1)}' + \ldots + \\
&Y_{(m+1)n}\,e_n' = -\left[Y_{(m+1)2}\,(e_1 - e_2) + \ldots + Y_{(m+1)m}\,(e_1 - e_m)\right] \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&(Y_{n1} + \ldots + Y_{nm})\,e_1' + Y_{n(m+1)}\,e_{(m+1)}' + \ldots + Y_{nn}\,e_n' \\
&\qquad = -\left[Y_{n2}\,(e_1 - e_2) + \ldots + Y_{nm}\,(e_1 - e_m)\right] \\
&(Y_{11} + \ldots + Y_{1m} + \ldots + Y_{mm})\,e_1' + (Y_{1(m+1)} + \ldots \\
&\qquad + Y_{m(m+1)})\,e_{(m+1)}' + \ldots + (Y_{1n} + \ldots + Y_{mn})\,e_n' \\
&\qquad = -\Big[(Y_{12} + \ldots + Y_{m2})\,(e_1 - e_2) + \ldots \\
&\qquad + (Y_{1m} + \ldots + Y_{mm})\,(e_1 - e_m) + \frac{e_1''}{2\,R}\Big]
\end{aligned}\right\} \tag{279}$$

These equations show that the solution for any of the reflected waves depends only on the incident waves $(e_1 \ldots e_m)$ on the ground wires, and are independent of the incident waves on the other conductors.

Of course, having found e_1' from (279), the reflected and refracted waves on the other ground wires follow

$$\left.\begin{aligned} e_1'' &= e_2'' = \ldots = e_m'' = e_1' + e_1 \\ e_2' &= e_2'' - e_2 = e_1' + e_1 - e_2 \\ &\ldots\ldots\ldots\ldots \\ e_m' &= e_m'' - e_m = e_1' + e_1 - e_m \end{aligned}\right\} \tag{280}$$

If there is only one ground wire, or if the m ground wires may be replaced by an *equivalent* ground wire in conformity with (166) and (171), then Equations (279) (putting $m = 1$) simplify to

$$\left.\begin{aligned} Y_{11}\, e_1' + \ldots + Y_{1n}\, e_n' &= -\,(e_1 + e_1')\; 2\,R \\ Y_{21}\, e_1' + \ldots + Y_{2n}\, e_n' &= 0 \\ &\ldots\ldots\ldots\ldots \\ Y_{n1}\, e_1' + \ldots + Y_{nn}\, e_n' &= 0 \end{aligned}\right\} \tag{281}$$

Solving for any e_k' there is

$$e_k' = -\frac{M_{1k}}{\begin{vmatrix} Y_{11} & \ldots & Y_{1n} \\ \cdot & \cdot & \cdot \\ Y_{n1} & \ldots & Y_{nn} \end{vmatrix}} \; \frac{(e_1 + e_1')}{2\,R} \tag{282}$$

Where M_{1k} is the minor whose cofactor is Y_{1k}. But by (162), (282) becomes

$$e_k' = -\frac{Z_{k1}}{2\,R}\,(e_1 + e_1') \tag{283}$$

For $k = 1$ this gives

$$e_1' = -\frac{Z_{11}}{2\,R + Z_{11}}\, e_1 = a\, e_1 \tag{284}$$

$$e_1'' = e_1 + e_1' = \frac{2\,R}{2\,R + Z_{11}}\, e_1 = b\, e_1 \tag{285}$$

$$e_k' = \frac{-Z_{k1}}{2\,R + Z_{11}}\, e_1 = c\, e_1 \tag{286}$$

$$e_k'' = e_k + e_k' = e_k + c\, e_1 \tag{287}$$

Periodic Resistance Grounding of Ground Wires.—Fig. 73 illustrates the several factors which determine the formation of induced traveling waves originating from the release of bound charges on the ground wires and on the line conductors. The ground wires are here shown grounded through the tower footing resistances. Just before the cloud begins to discharge, the electrostatic field gradient G is constant; and, proportional to its distribution, bound charges $Q_1 f(x)$ and $Q_2 f(x)$ reside on the ground wires and line conductors. These bound charges may be computed from Equations (252). Now, as the cloud discharges according to some law $F(t)$, the field gradient diminishes accordingly, and elements of the bound charges are released as traveling-wave components moving away in opposite directions. When waves on the ground wires encounter the towers, reflections occur on both the ground wires and line wires. These may be calculated by Equations (284), (285), (286), and (287) if there is only one ground wire. If there are several ground wires at the same level, then they may be replaced by a single equivalent ground wire in conformity with the requirements of Equations (166) and (171). But if the ground wires are unsymmetrically placed so that they are not at the same potential (prior to reflection), Equations (279) apply, but they will not be invoked in the present analysis.

The induced surge may be calculated in the following steps:

a. Calculate the bound charges from (252), and consider each span separately. The total bound charge on any wire is then the sum of the bound charges for each span, and each of these may be handled separately.

b. Replace the actual ground wires by an equivalent ground wire consistent with the conditions imposed by (166) and (171).

c. Compute the traveling waves on every wire for each span. The shape of these traveling waves is given by (257) or its graphical or tabular method counterpart.

d. Calculate the reflection and refraction operators for both line and ground wires, as given by (284) to (287). Then

$$
\begin{aligned}
e_1' &= a\, e_1 && = \textit{reflected wave on ground wire}\\
e_1'' &= b\, e_1 && = \textit{transmitted wave on ground wire}\\
e_2' &= c\, e_1 && = \textit{reflected wave on line wire}\\
e_2'' &= c\, e_1 + e_2 && = \textit{transmitted wave on line wire}
\end{aligned}
$$

e. Construct a lattice as shown in Fig. 75 of a sufficient number of sections to include the requisite time interval and number of spans, and therefrom determine the potentials at all points and times by superposition of the waves from the bound charges of all spans.

The simplest case is that of instantaneous cloud discharge and

Of course, having found e_1' from (279), the reflected and refracted waves on the other ground wires follow

$$\left.\begin{array}{l} e_1'' = e_2'' = \ldots = e_m'' = e_1' + e_1 \\ e_2' = e_2'' - e_2 = e_1' + e_1 - e_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ e_m' = e_m'' - e_m = e_1' + e_1 - e_m \end{array}\right\} \tag{280}$$

If there is only one ground wire, or if the m ground wires may be replaced by an *equivalent* ground wire in conformity with (166) and (171), then Equations (279) (putting $m = 1$) simplify to

$$\left.\begin{array}{l} Y_{11}\, e_1' + \ldots + Y_{1n}\, e_n' = -\,(e_1 + e_1')\; 2\,R \\ Y_{21}\, e_1' + \ldots + Y_{2n}\, e_n' = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Y_{n1}\, e_1' + \ldots + Y_{nn}\, e_n' = 0 \end{array}\right\} \tag{281}$$

Solving for any e_k' there is

$$e_k' = - \left| \frac{M_{1k}}{\begin{array}{ccc} Y_{11} & \ldots & Y_{1n} \\ \cdot & \cdot & \cdot \\ Y_{n1} & \ldots & Y_{nn} \end{array}} \right| \frac{(e_1 + e_1')}{2\,R} \tag{282}$$

Where M_{1k} is the minor whose cofactor is Y_{1k}. But by (162), (282) becomes

$$e_k' = - \frac{Z_{k1}}{2\,R}\,(e_1 + e_1') \tag{283}$$

For $k = 1$ this gives

$$e_1' = - \frac{Z_{11}}{2\,R + Z_{11}}\, e_1 = a\, e_1 \tag{284}$$

$$e_1'' = e_1 + e_1' = \frac{2\,R}{2\,R + Z_{11}}\, e_1 = b\, e_1 \tag{285}$$

$$e_k' = \frac{-Z_{k1}}{2\,R + Z_{11}}\, e_1 = c\, e_1 \tag{286}$$

$$e_k'' = e_k + e_k' = e_k + c\, e_1 \tag{287}$$

Periodic Resistance Grounding of Ground Wires.—Fig. 73 illustrates the several factors which determine the formation of induced traveling waves originating from the release of bound charges on the ground wires and on the line conductors. The ground wires are here shown grounded through the tower footing resistances. Just before the cloud begins to discharge, the electrostatic field gradient G is constant; and, proportional to its distribution, bound charges $Q_1 f(x)$ and $Q_2 f(x)$ reside on the ground wires and line conductors. These bound charges may be computed from Equations (252). Now, as the cloud discharges according to some law $F(t)$, the field gradient diminishes accordingly, and elements of the bound charges are released as traveling-wave components moving away in opposite directions. When waves on the ground wires encounter the towers, reflections occur on both the ground wires and line wires. These may be calculated by Equations (284), (285), (286), and (287) if there is only one ground wire. If there are several ground wires at the same level, then they may be replaced by a single equivalent ground wire in conformity with the requirements of Equations (166) and (171). But if the ground wires are unsymmetrically placed so that they are not at the same potential (prior to reflection), Equations (279) apply, but they will not be invoked in the present analysis.

The induced surge may be calculated in the following steps:

a. Calculate the bound charges from (252), and consider each span separately. The total bound charge on any wire is then the sum of the bound charges for each span, and each of these may be handled separately.

b. Replace the actual ground wires by an equivalent ground wire consistent with the conditions imposed by (166) and (171).

c. Compute the traveling waves on every wire for each span. The shape of these traveling waves is given by (257) or its graphical or tabular method counterpart.

d. Calculate the reflection and refraction operators for both line and ground wires, as given by (284) to (287). Then

$$\begin{aligned}
e_1' &= a\, e_1 && = \textit{reflected wave on ground wire}\\
e_1'' &= b\, e_1 && = \textit{transmitted wave on ground wire}\\
e_2' &= c\, e_1 && = \textit{reflected wave on line wire}\\
e_2'' &= c\, e_1 + e_2 && = \textit{transmitted wave on line wire}
\end{aligned}$$

e. Construct a lattice as shown in Fig. 75 of a sufficient number of sections to include the requisite time interval and number of spans, and therefrom determine the potentials at all points and times by superposition of the waves from the bound charges of all spans.

The simplest case is that of instantaneous cloud discharge and rectangular bound charges. This case also gives the maximum departure of the potential at midspan with respect to that at the towers. Curves calculated on these assumptions for a bound charge 2000 ft. long and 1000-ft. spans are shown in Fig. 76. The traveling wave on the line wire is 0.550 (18 per cent high) at $t = 0$, 0.488 (4 per cent high) at $t = 1$, and 0.472 (1 per cent high) at $t = 2$ ms., respectively. Had the ground wires been *ideal* the voltage would have been 0.467—a value quickly approached by successive reflections.

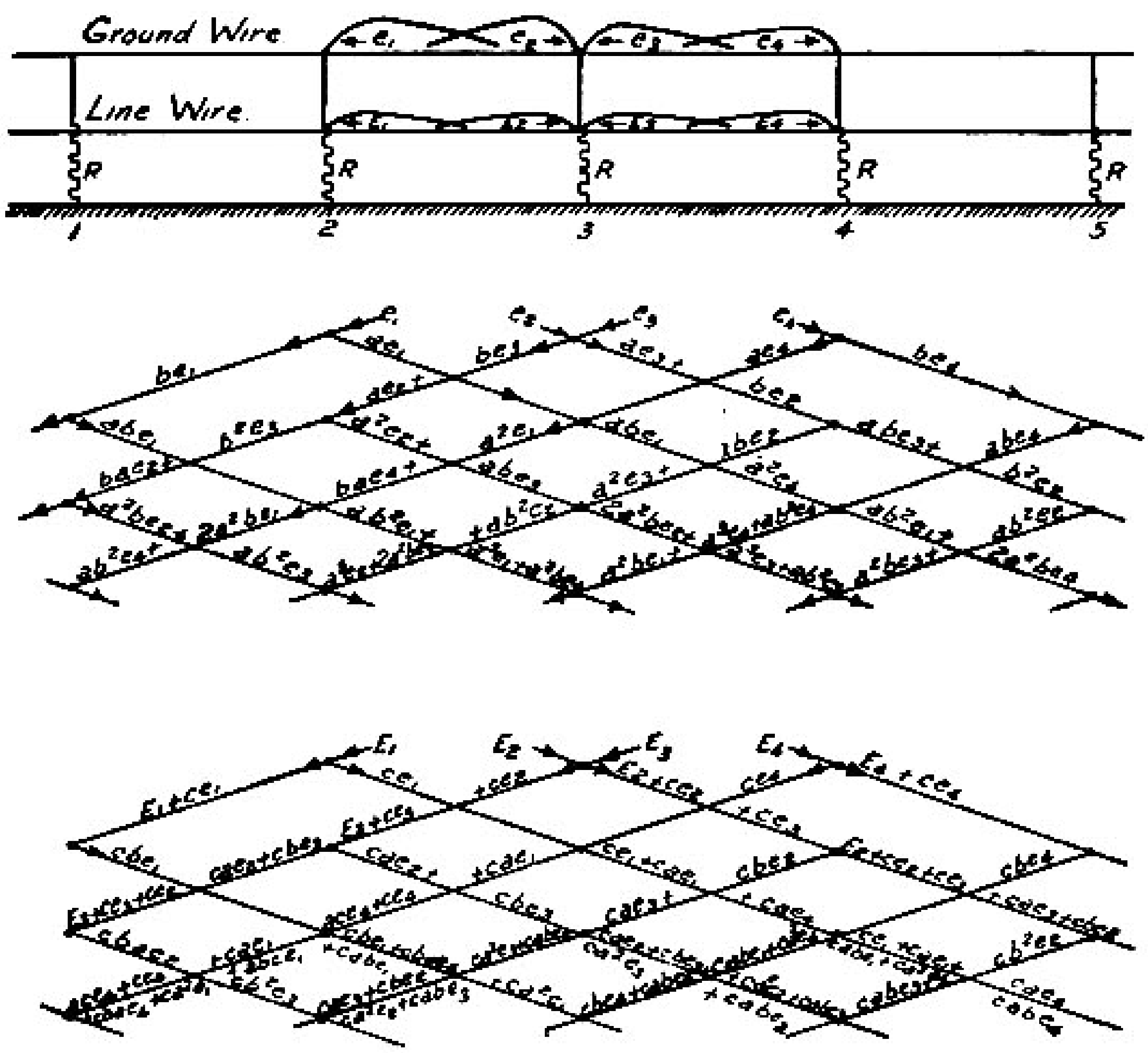


FIG. 75.—Waves from Released Bound Charges

Top lattice: Reflections and refractions on ground wire. Bottom lattice: Reflections and refractions on line wire

The potential difference between the ground and line wires is much greater for a short period (1.30 for 1/2 ms.). However, under the actual conditions of a finite rate of cloud discharge, this difference would only be slightly in excess of 0.467. There is no need, therefore, to complicate the calculation of the protective ratio beyond that required for ideal ground wires.

Direct Strokes.—The investigation of direct lightning strokes on transmission lines falls into two categories. First, there are the statistical studies relating to the probability of a line being struck, the magnitude of the voltages and currents involved, and the duration of the surge; and second, there is the analysis of the resulting disturbances.

A considerable amount of information concerning the paths and frequency of storms throughout the United States is available from the meteorological survey charts issued by the Bureau of Standards.

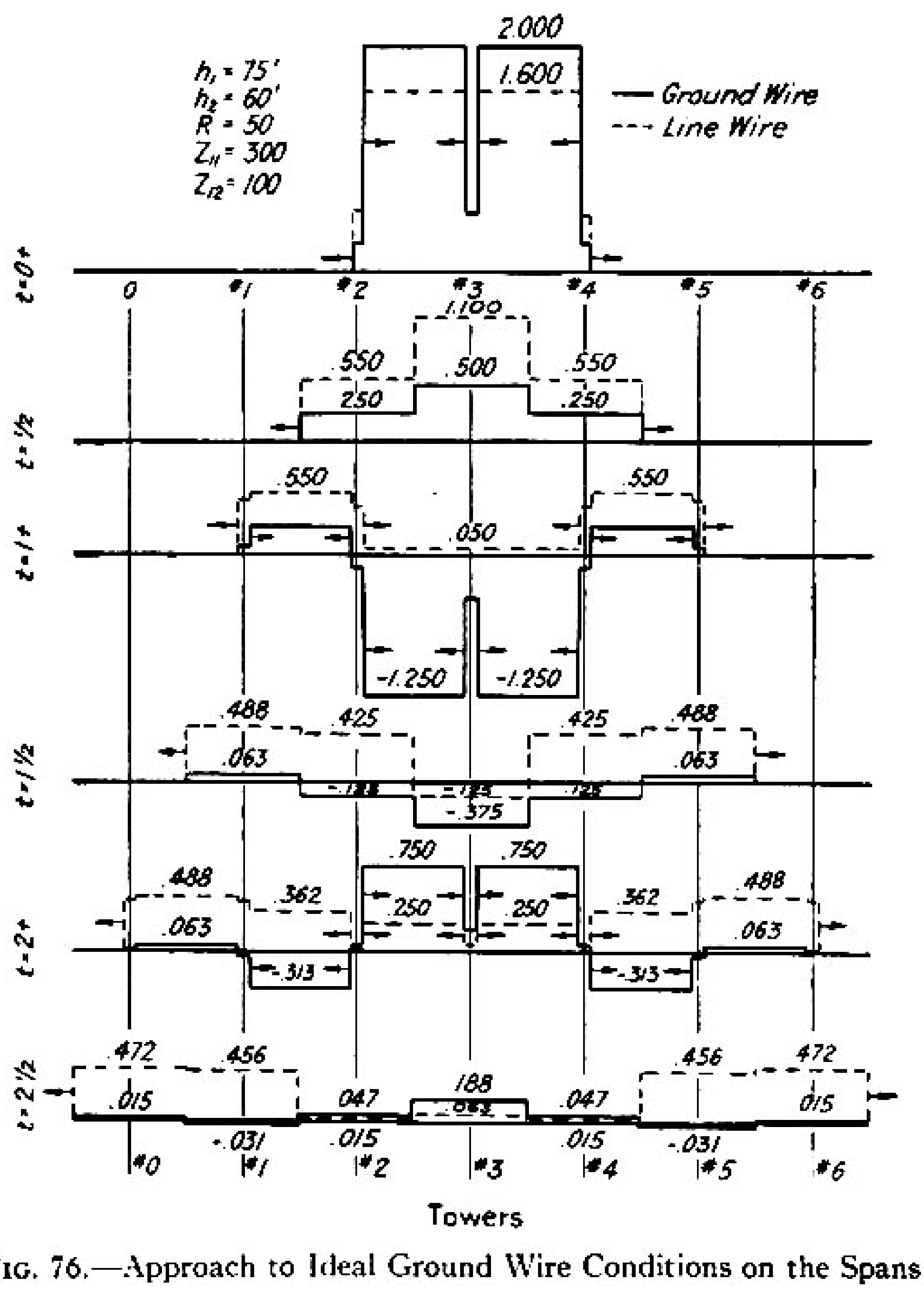


FIG. 76.—Approach to Ideal Ground Wire Conditions on the Spans

These data are further supplemented by the operating records of numerous power companies. Thus rough estimates may be made of the number of storms per unit area per year in a given locality.

In Fig. 77*A* is shown a transmission line tower on a level plane free of brush or other projections. Although the idiosyncrasies of a lightning bolt are too erratic for anyone to predict where it will go, yet the only justifiable assumption is to suppose that, on the average, it will strike to the nearest object. On this basis, the stroke will, by preference, hit the tower, when the distance r is less than the height H. In Fig. 77*B* is plotted a set of curves showing the distance D, corresponding to $r = H$, between the projection of the approaching center of disturbance and the tower, as function of the cloud height H and the tower height h. If there is absolute certainty that lightning will strike within a zone of width W centered on the transmission line, then the probability that the line will receive the stroke is $2\ D/W$.

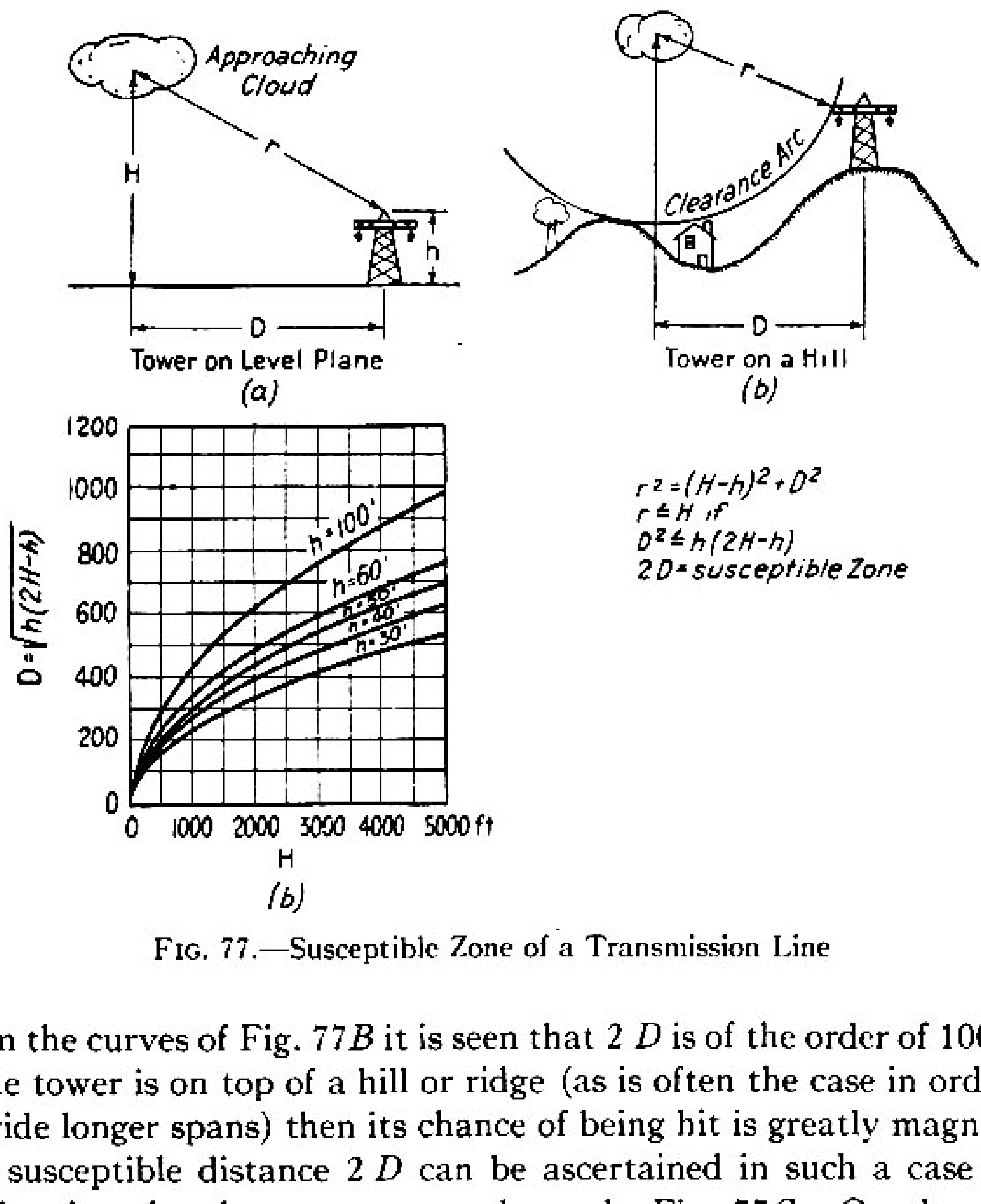


FIG. 77.—Susceptible Zone of a Transmission Line

From the curves of Fig. 77*B* it is seen that $2\ D$ is of the order of 1000 ft. If the tower is on top of a hill or ridge (as is often the case in order to provide longer spans) then its chance of being hit is greatly magnified. The susceptible distance $2\ D$ can be ascertained in such a case only by drawing the clearance arc as shown in Fig. 77*C*. On the other hand, the proximity of trees or other nearby projections decreases the chance of a direct hit to the line. The tower is more likely to be hit than the sagging span, because it is higher.

It has been proposed to provide line towers with extension masts of sufficient height to insure their being struck instead of any part of the line. The necessary height of such masts is of the order of 300 ft. for 1000-ft. spans. Still other schemes have been advanced, such as paralleling the transmission line by separate masts supporting a direct-hit wire.

The voltage of a lightning stroke has never been directly measured, but calculations based on current measurements suggest that the maximum is from ten to fifteen million volts at the point of contact with the transmission line.

The ground wire should be located high enough above the line conductors so that it adequately shields them from the direct stroke. Curves applying to 1000-ft. cloud heights are given in Fig. 78, from which the necessary ground-wire height can be determined.

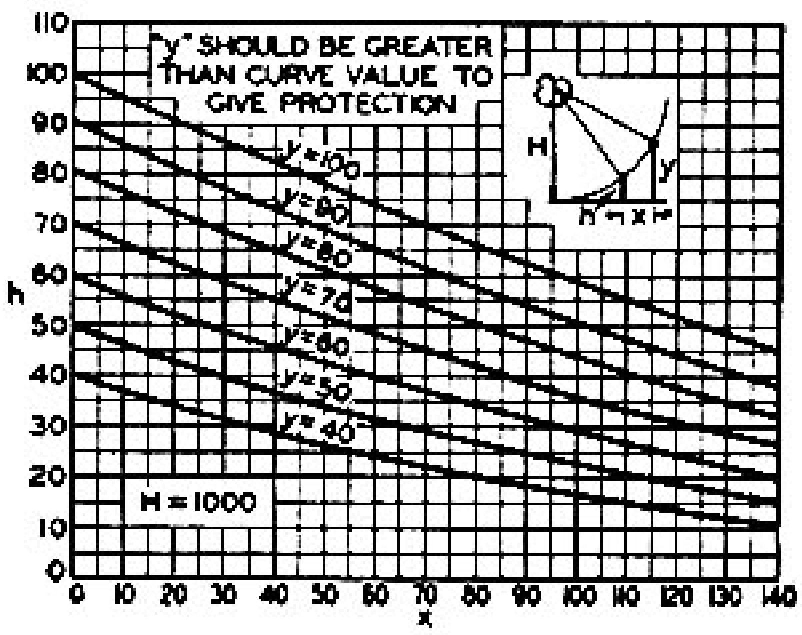


FIG. 78.—Location of Ground Wire to Take Initial Hit

For higher clouds there is a greater margin of safety. The equation of these curves is

$$\frac{x}{H} = \sqrt{2\left(\frac{y}{H}\right) - \left(\frac{y}{H}\right)^2} - \sqrt{2\left(\frac{h}{H}\right) - \left(\frac{h}{H}\right)^2} \qquad (288)$$

where x, y, h, and H are defined in Fig. 78. Almost invariably, if the

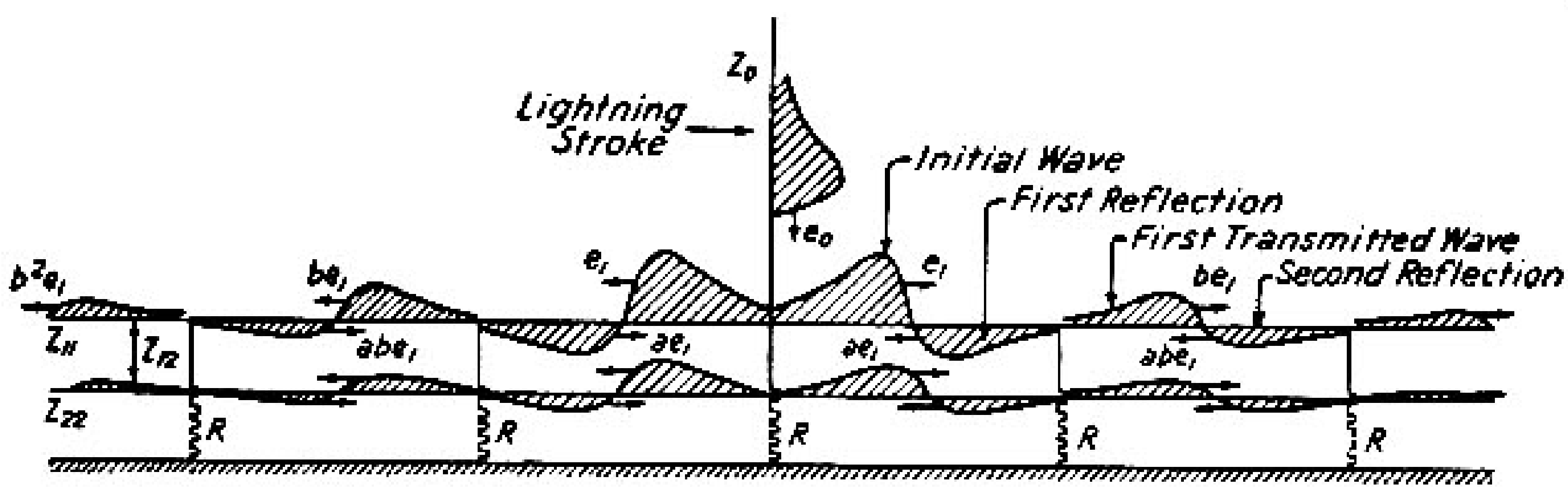


FIG. 79.—Reflected and Transmitted Waves Due to Periodic Resistance Grounding

ground wires are located far enough above the line wires to prevent sparkover between them, the clearance will be more than sufficient for shielding purposes.

When lightning strikes the tower (No. 1), Fig. 79, a pair of traveling waves e_1 start out on the ground wires accompanied by companion waves on the line wires.

Let Z_0 = surge impedance of the lightning stroke.

Z_{11} = self surge impedance of the equivalent ground wire.

Z_{1k} = mutual surge impedance between the equivalent ground wire and any line wire k.

e_0 = incident wave of the lightning stroke.

e_1 = initial wave on the ground wire.

R = tower footing resistance.

I = tower current.

Since there can be no current in the line wires at the point of stroke, it follows that

$$e_0 + e_0' = R\,I = e_1$$

$$\left(\frac{e_0 - e_0'}{Z_0}\right) = I + 2\,i_1 = I + 2\left(\frac{e_1}{Z_{11}}\right)$$

$$e_k = Z_{1k}\,i_1$$

Therefore

$$e_1 = \frac{2\,R\,Z_{11}\,e_0}{Z_{11}\,R + Z_0\,Z_{11} + 2\,Z_0\,R} \tag{289}$$

$$e_k = \frac{Z_{1k}}{Z_{11}}\,e_1 \tag{290}$$

When these waves reach the adjacent towers they are reflected and refracted in accordance with Equations (284) to (287). Waves arriving at Tower 1 from the reflection points are reflected therefrom as

$$e_1' = \frac{2\,Z_0\,R - Z_{11}\,(Z_0 + R)}{2\,Z_0\,R + Z_{11}\,(Z_0 + R)}\,e_1 = \alpha\,e_1 \tag{291}$$

$$e_k' = e_k + \frac{-2\,Z_{1k}\,(R + Z_0)}{2\,Z_0\,R + Z_{11}\,(Z_0 + R)}\,e_1 = e_k + \beta\,e_1 \tag{292}$$

These two equations follow from (284) to (287) upon superimposing the two waves reaching Tower 1 simultaneously from both the right and left, regarding R and Z_0 in parallel.

Inspection of (289), (290), (291), and (292) shows that these equations also result if the lightning bolt is assumed to have a surge impedance of $2\,Z_0$ and if the line and ground wires extend in one direction only from Tower 1 as shown in Fig. 80. Then the reflection lattices of Figs. 80 and 81 may be constructed. The potential wave passing from the lightning bolt to the ground wire is e_1. When this wave reaches Tower 2 a part $a\,e_1$, Equation (284), is reflected back, and a part $b\,e_1$, Equation (285), is passed on. Likewise, on the line wire, $c\, e_1$, Equation (286), is reflected, while $(e_k + c\, e_1)$, Equation (287), is transmitted. The same thing happens at all towers, except No. 1, where the reflections obey (291) and (292). By means of these lattices the potential at any tower at any time may be readily calculated.

In Fig. 82 are shown the potentials on the ground and line wires and across the insulators, as function of the tower footing resistance,

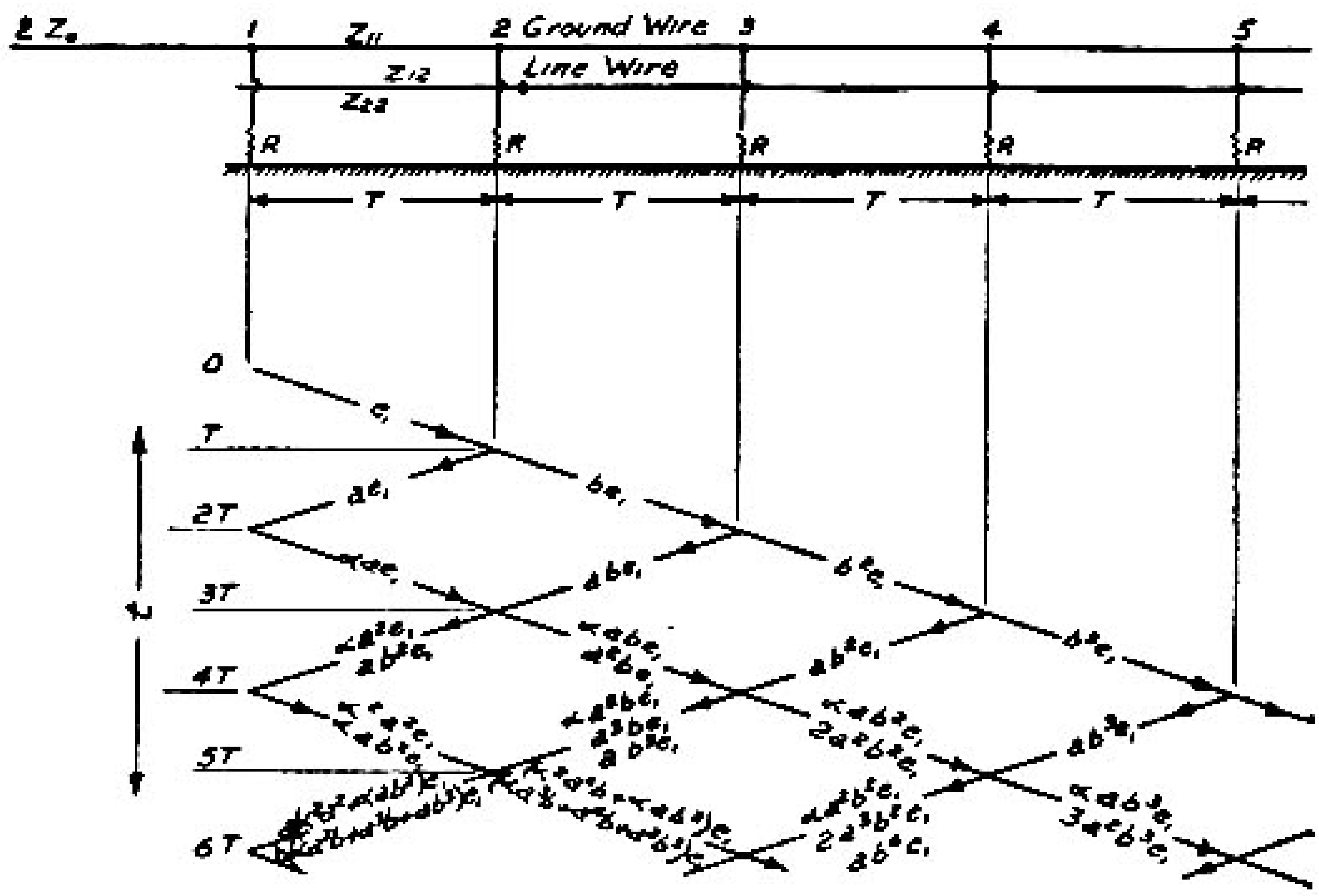


FIG. 80.—Reflections and Refractions on Ground Wire. Lightning Stroke at Tower

The voltages on the ground wire at the junctions as functions of time are:

$$e_1 = e_1(t) + [a + \alpha\, a]\cdot e_1(t - 2T) + [(1 + \alpha)(\alpha\, a^2 + a\, b^2)]\cdot e_1(t - 4T) + \ldots$$
$$e_2 = b\, e_1(t - T) + [a\, b + a^2 b + \alpha\, a\, b]\cdot e_1(t - 3T) + \ldots.$$
$$e_3 = b^2 e_1(t - 2T) + [a\, b^2 + 2\, a^2 b^2 + \alpha\, a\, b^2)\cdot e_1(t - 4T) + \ldots.$$
$$e_4 = b^3 e_1(t - 3T) + \ldots.$$

and for several towers. It is thus evident that low tower footing resistance has five beneficial results:

1. Reduced potentials on the ground wires.
2. Reduced potentials on the line wires.
3. Reduced potentials across the insulators.
4. Limitation of the disturbance to a few spans.
5. Shorter duration of dangerous voltages.

For tower footing resistances below 25 ohms the reduction in voltages is practically directly proportional to the reduction in resistance; that is, 10 ohms allow twice the voltages that 5 ohms would permit. It is also to be noticed that the disturbance resulting from a direct lightning stroke which involves the ground wire is not a simple traveling wave free to continue its travel along the line, but is a great multiplicity of successive reflections which are necessarily limited to a few spans, and are rapidly dissipated. Therefore, unless the lightning bolt contacts a line conductor, station apparatus is comparatively safe from direct strokes more than, say, ten towers away. It is recommended that every precaution be taken to reduce the tower footing resistance to about 5 ohms on the first few towers from the

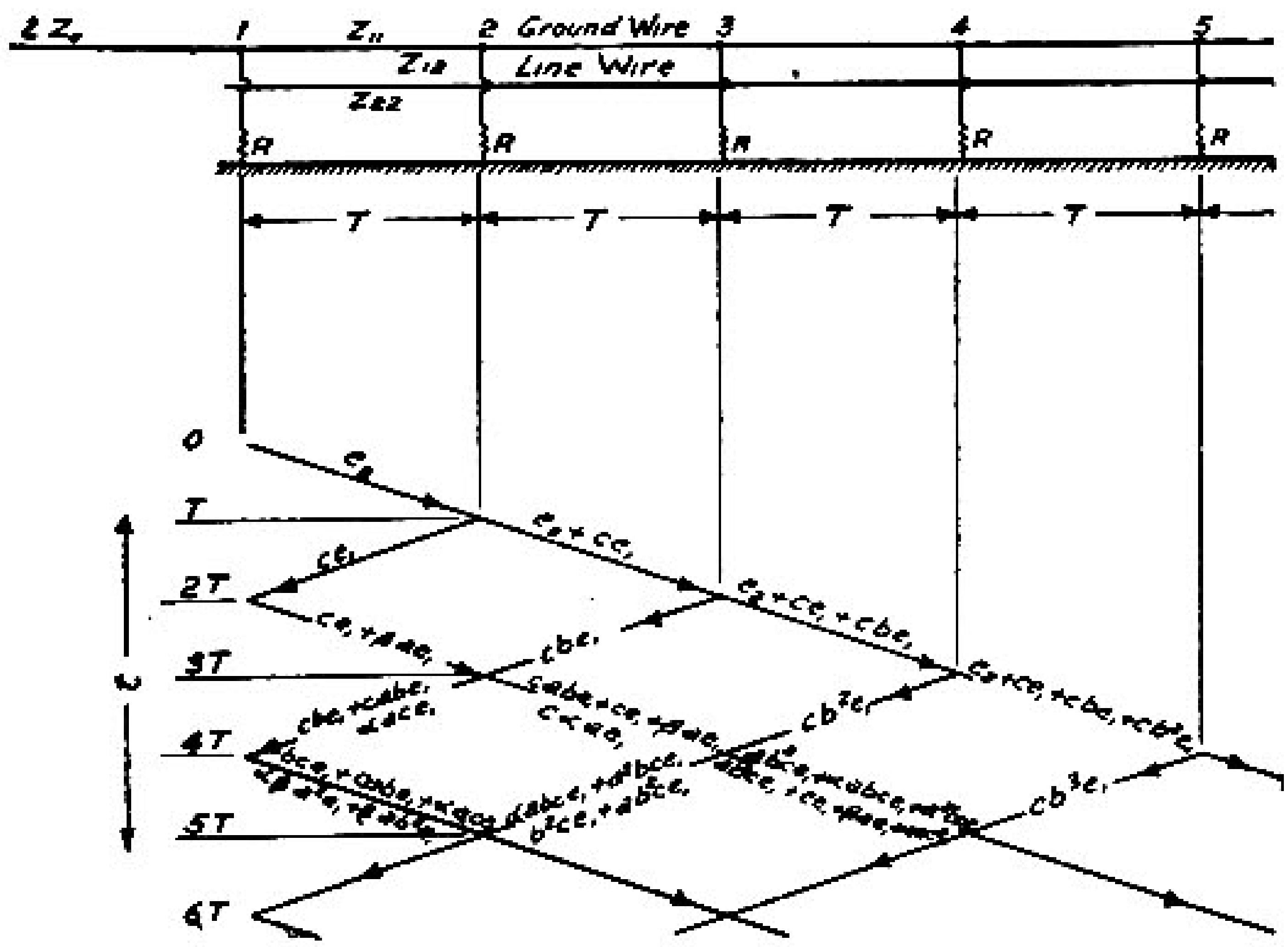


FIG. 81.—Reflections and Refractions on Line Wire. Lightning Stroke at Tower

The voltages on the line wire at the junctions as functions of time are:

$$E_1 = e_2(t) + [2c + \beta a]\cdot e_1(t - 2T) + [2bc + 2abc + 2\alpha ac + \alpha\beta a^2 + \beta a b^2]\cdot e_1(t - 4T) + \ldots.$$

$$E_2 = e_2(t - T) + c\,e_1(t - T) + [c + \beta a + bc + abc + \alpha ac]\cdot e_1(t - 3T) + \ldots.$$

$$E_3 = e_2(t - 2T) + [c + bc]\cdot e_1(t - 2T) + [b^2c + ab^2c + \alpha abc + a^2bc + abc + c + \beta a + \alpha ac]\cdot e_1(t - 4T) + \ldots.$$

$$E_4 = e_2(t - 3T) + [c + bc + b^2c]\cdot e_1(t - 3T) + \ldots.$$

station. It is also advisable to install extra ground wires on this section to shield the line conductors more effectively.

Fig. 83 shows the effect of the lightning surge wave shape on the potentials on the first three towers. The longer the front of the wave the lower the voltages, for the reflections are able to start reduction before the initial wave is fully developed.

Instruments have been installed on towers for measuring the currents due to a direct stroke. The sum of the currents measured on all adjacent towers from a single lightning stroke will exceed the actual current in the stroke, because, as is evident from Fig. 80, the maximum currents in different towers occur at different instants. The maximum voltage on any tower can be found from Fig. 80, and the maximum current is simply

$$I = \frac{e_{max}}{R}$$

The ratio of the sum of the maximum tower currents to the maximum current in the stroke is given in Fig. 84 as function of the tower footing resistance. Thus for $R = 60$ the sum of the measured values may exceed the actual value by 40 per cent.

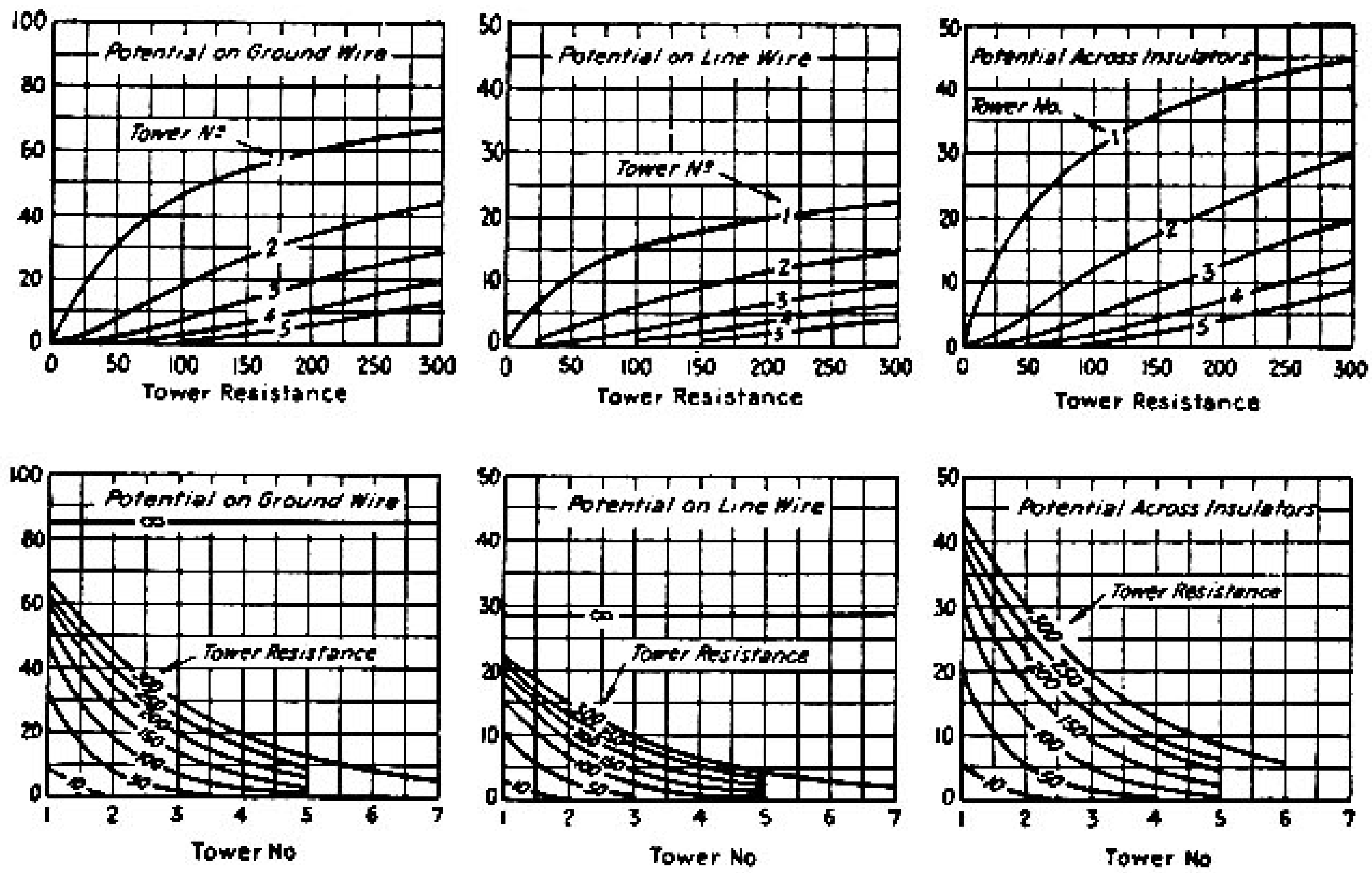


FIG. 82.—Potentials at the Towers in Percent of the Lightning Voltage

If lightning strikes the ground wire at midspan ($R = \infty$), and flashover does not take place, there is by (289) and (290)

$$e_1 = \frac{2\, Z_{11}\, e_0}{Z_{11} + 2\, Z_0} \tag{293}$$

$$e_k = \frac{Z_{1k}}{Z_{11}}\, e_1 \tag{294}$$

$$e_1 - e_k = \left(1 - \frac{Z_{1k}}{Z_{11}}\right) e_1 \tag{295}$$

These voltages will then persist until reduced by the reflections from the towers. If the " length " of the span in microseconds is T, then that time must elapse before relief arrives. In the meantime, flash-over between ground wire and line conductor should not occur. This will not happen if the separation is such that the sparkover voltage is not reached before the waves of reduction return from the towers.

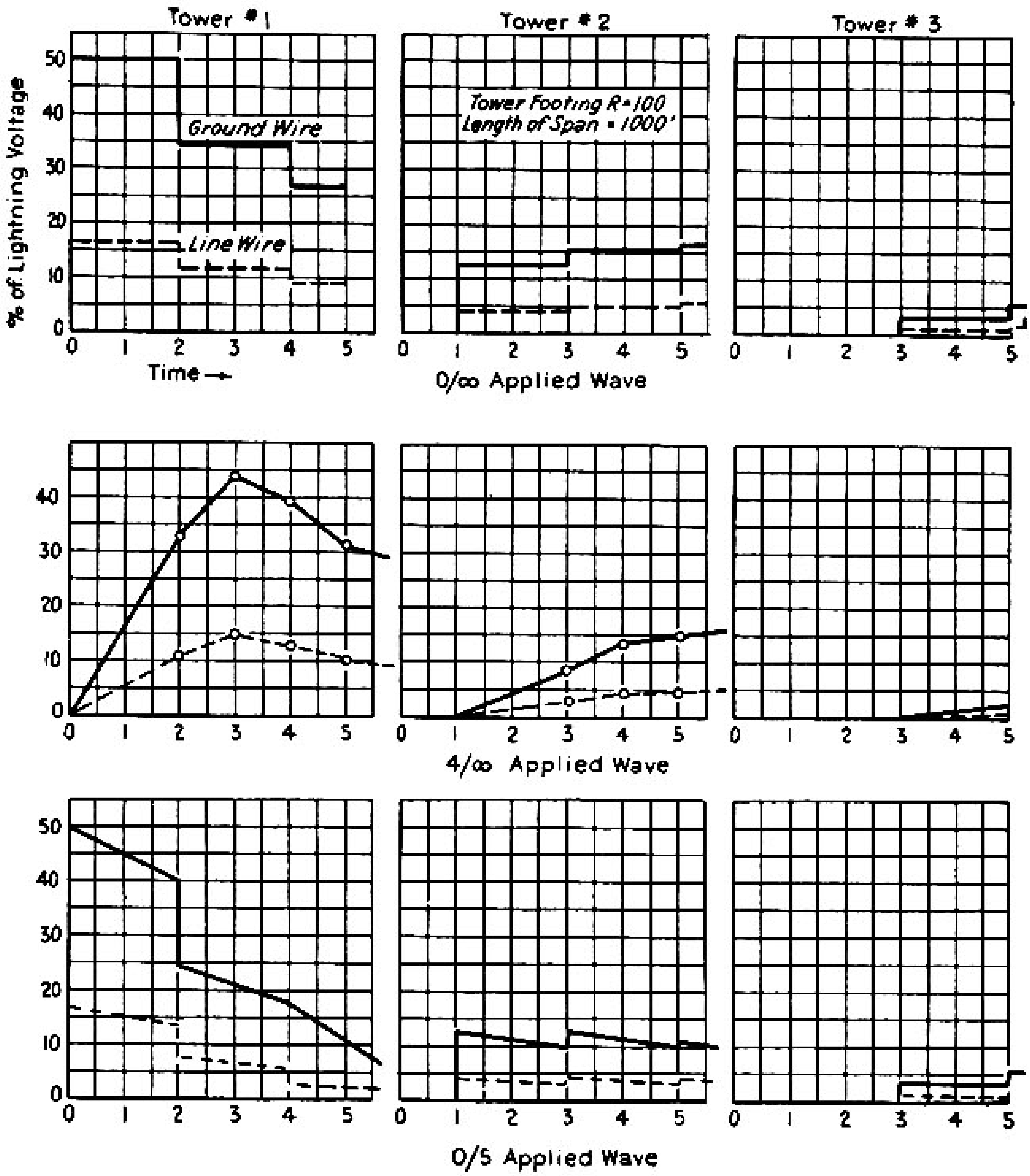


FIG. 83.—Potentials at the Towers as Functions of Time

In Fig. 85 the voltages on the ground wire, line conductor, and between them have been plotted as function of the separation (the mutual surge impedance Z_{1k} decreases with the separation); and the sparkover characteristics are given for different time lags. It is seen that for 1000-ft. spans the separation at midspan should be at least 28 ft.

By increasing the tension on the ground wires it is usually possible to obtain the required separation at midspan with smaller separation at the towers.

The following table gives the voltages due to a direct stroke either at the tower or at midspan.

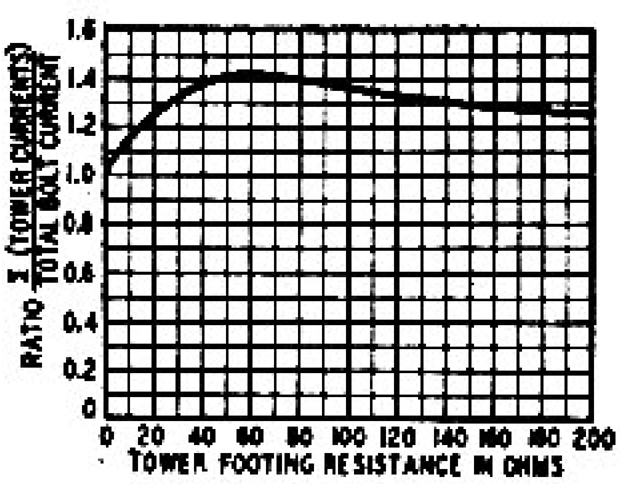


FIG. 84.—Calculated Relation of Sum of Tower Currents to Total Current in Lightning Bolt for Tower with Different Footing Resistances

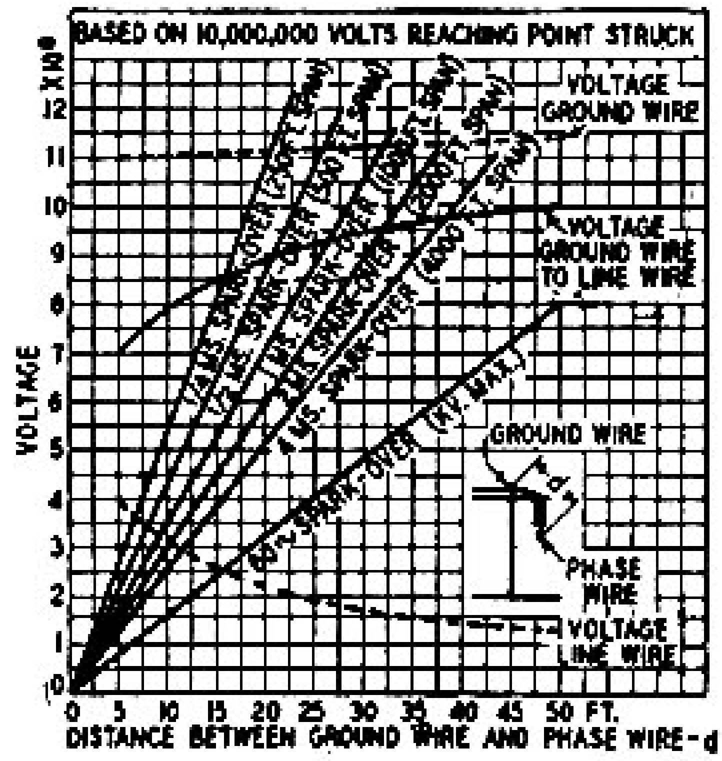


FIG. 85.—Minimum Clearance Necessary to Prevent Sparkover from Ground Wire to Line for a Direct Hit to Ground Wire in Center of Span

Stroke at Tower	Voltages	Stroke at Midspan
$\dfrac{2Rz_{11}E}{(z_{11}+2z_0)R+z_{11}z_0}$ . . .	$V_{g.w.}$(tower) . . .	$\left(\dfrac{2R}{2R+z_{11}}\right)\left(\dfrac{2z_{11}}{z_{11}+2z_0}\right)E$
$\dfrac{2Rz_{11}E}{(z_{11}+2z_0)R+z_{11}z_0}$ . . .	$V_{g.w.}$(midspan) . . .	$\left(\dfrac{2z_{11}}{z_{11}+2z_0}\right)E$
$\dfrac{2Rz_{1r}E}{(z_{11}+2z_0)R+z_{11}z_0}$ . . .	V_{line}(tower) . . .	$\left(\dfrac{2R}{2R+z_{11}}\right)\left(\dfrac{2z_{1r}}{z_{11}+2z_0}\right)E$
$\dfrac{2Rz_{1r}E}{(z_{11}+2z_0)R+z_{11}z_0}$ . . .	V_{line}(midspan) . . .	$\left(\dfrac{2z_{1r}}{z_{11}+2z_0}\right)E$
$\dfrac{2R(z_{11}-z_{1r})E}{(z_{11}+2z_0)R+z_{11}z_0}$ . . .	$V_{insul.}$(tower) . . .	$\left(\dfrac{4R}{2R+z_{11}}\right)\left(\dfrac{z_{11}-z_{1r}}{z_{11}+2z_0}\right)E$
$\dfrac{2R(z_{11}-z_{1r})E}{(z_{11}+2z_0)R+z_{11}z_0}$ . . .	$V_{insul.}$(midspan) . . .	$2\left(\dfrac{z_{11}-z_{1r}}{z_{11}+2z_0}\right)E$

z_{11} = equivalent self surge impedance of all ground wires.

z_{1r} = equivalent mutual surge impedance of all ground wires to any line wire r.

z_0 = surge impedance of lightning bolt.
R = tower footing resistance.
E = voltage of incident wave from lightning stroke.

In terms of the current in the tower, the voltages (for a strike at the tower) are

$$V_{\text{g.w.}} = V_{\text{tower}} = R\,I$$

$$V_{\text{line}} = R\,I\left(\frac{z_{1r}}{z_{11}}\right)$$

$$V_{\text{insul.}} = R\,I\left(\frac{z_{11} - z_{1r}}{z_{11}}\right)$$

Ordinarily

$$z_{1r}\ z_{11} \cong 0.20$$

so that

$$V_{\text{insul.}} \cong 0.80\,R\,I$$

Introduction of Extra Ground Wires.—It has often been suggested that the first few towers out from the station should be provided with extra ground wires. The arguments favoring such an installation are:

1. Greater shielding effect from direct strokes.
2. Lower induced surges on that section.
3. Reduction of incoming waves due to a reduced surge impedance.

The last advantage is not of much importance, since the reduction is only a few per cent. The transition-point equations for the n line wires at the point where the extra ground wires are introduced are

$$\left.\begin{array}{c} Y_{11}(e_1 - e_1') + \dots + Y_{1n}(e_n - e_n') = y_{11} e_1'' + \dots + y_{1n} e_n'' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_{n1}(e_1 - e_1') + \dots + Y_{nn}(e_n - e_n') = y_{n1} e_1'' + \dots + y_{nn} e_n'' \\ e_1 + e_1' = e_1'' \\ \cdots \\ e_n + e_n' = e_n'' \end{array}\right\} \quad (296)$$

Rearranging

$$\left.\begin{array}{l} (Y_{11} + y_{11})\, e_1'' + \dots + (Y_{1n} + y_{1n})\, e_n'' \\ \qquad = 2\,(Y_{11}\, e_1 + \dots + Y_{1n}\, e_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (Y_{n1} + y_{n1})\, e_1'' + \dots + (Y_{nn} + y_{nn})\, e_n'' \\ \qquad = 2\,(Y_{n1}\, e_1 + \dots + Y_{nn}\, e_n) \end{array}\right\} \quad (297)$$

Herefrom any transmitted wave e_k'' may be calculated. The surge admittances, of course, involve the ground wires. As an example, consider a single line conductor and one ground wire entering a section over which there are two ground wires. Let

$$Z_{11} = 450 \qquad z_{11} = 450 \qquad z_{12} = 100$$
$$Z_{22} = 500 \qquad z_{22} = 500 \qquad z_{23} = 100$$
$$Z_{12} = 100 \qquad z_{33} = 500 \qquad z_{31} = 100$$

Then by (161)

$$Y_{11} = 0.002325 \quad \text{and} \quad y_{11} = 0.00240$$

By (297) the transmitted wave on the line conductor is

$$e_1'' = \frac{2\, Y_{11}\, e_1}{Y_{11} + y_{11}} = \frac{0.004650}{0.004725}\, e_1 = 0.983\, e_1$$

Thus the reduction is only 2 per cent. Of course, by using more extra ground wires and placing them closer to the line conductors, a greater reduction can be secured, but it is difficult to get more than 5 per cent reduction.

Grounding Rods and Earth Wires.—The resistance of driven grounds depends upon the resistivity ρ of the earth, the diameter $2\,r$ and length L of the ground rod, and the number N of rods and the spacing s between them. H. B. Dwight * gives for the resistance of a single rod

$$R_1 = \frac{\rho}{2\,\pi L}\left(\log\frac{4\,L}{r} - 1\right)$$

and for a pair of rods separated by s centimeters

$$R_2 = \frac{\rho}{4\,\pi L}\left\{\left[\log\frac{4\,L}{r} - 1\right] + \left[\log\frac{2\,L + \sqrt{s^2 + 4\,L^2}}{s} + \frac{s}{2\,L} - \frac{\sqrt{s^2 + 4\,L^2}}{2\,L}\right]\right\}$$

and for N ground rods, approximately

$$R_n \cong \frac{\rho}{2\,\pi LN}\left(\log\frac{4\,L}{r} - 1\right) + \frac{\rho}{2\,\pi LN} \sum\left[\log\frac{2\,L + \sqrt{s^2 + 4\,L^2}}{s} + \frac{s}{2\,L} - \frac{\sqrt{s^2 + 4\,L^2}}{2\,L}\right]$$

* "The Calculation of Resistances to Ground and of Capacitance," *Journal of Mathematics and Physics*, Vol. X, No. 1, 1931.

where the summation is to include the s distances from the central rod to all other rods.

The resistivity of the earth varies over a wide range and is not constant with respect to voltage. At low values the ground resistance to impulses is practically the same as that measured by direct current. But for higher values the resistance to high-voltage impulses is always less, the more so the higher the d-c. resistance and the higher the voltage. Consequently, measured values of ground resistance are merely an indication of the order of magnitude. Calculated values are of still less importance, because such calculations assume an earth of homogeneous constant resistivity, whereas the resistivity varies not only with the applied impulse voltage, but also with the moisture content of the soil, and with the depth. Ordinarily, the rods are 8 or 10 ft. long, and if more than one are used they are spaced about 10 ft. apart. The following table is indicative of the gain that may be realized in fairly wet soil by paralleling several rods:*

Number of rods	1	2	3	4	5	6	7	8
Ohms	40	22	14	10	7	5	3.5	2.5

For more details, including curves, the reader should consult the articles referred to.

A wire laid on, or buried beneath, the surface of the ground, and connected to the tower footing, is called a counterpoise, or earth wire. These counterpoises may prove quite efficient in reducing the resistance. In effect, they behave as short and very leaky transmission lines. Since the reflections on such a leaky line subside in one or two oscillations, the counterpoise very quickly acts as a series resistance with distributed leakage to ground, as shown in Fig. 86. The effective terminal resistance is

$$R_2 = \sqrt{\frac{r}{G}} \coth \sqrt{r\,G}\,l$$

which is in parallel with the tower footing resistance R_1, so that the net resistance is

$$R = \frac{R_1\,R_2}{R_1 + R_2} = \frac{R_1}{1 + R_1\sqrt{\frac{G}{r}}\tanh \sqrt{r\,G}\,l}$$

* "Lightning Arrester Grounds," by H. M. Towne, *General Electric Review*, March, April, May, 1932.

In Fig. 86 this equation has been fitted to two different test curves. It is seen that the agreement is quite good. The major gain is realized in the first two or three hundred feet. This suggests that a given length of wire is best employed as several 200 300 lengths rather than as one long wire.

Summing up, the design of the lightning protection for a transmission line resolves into the following considerations:

a. Consult the meteorological charts and the records of power companies operating in the same territory, to get an idea of the number, severity, and season of lightning storms in the locality of the line.

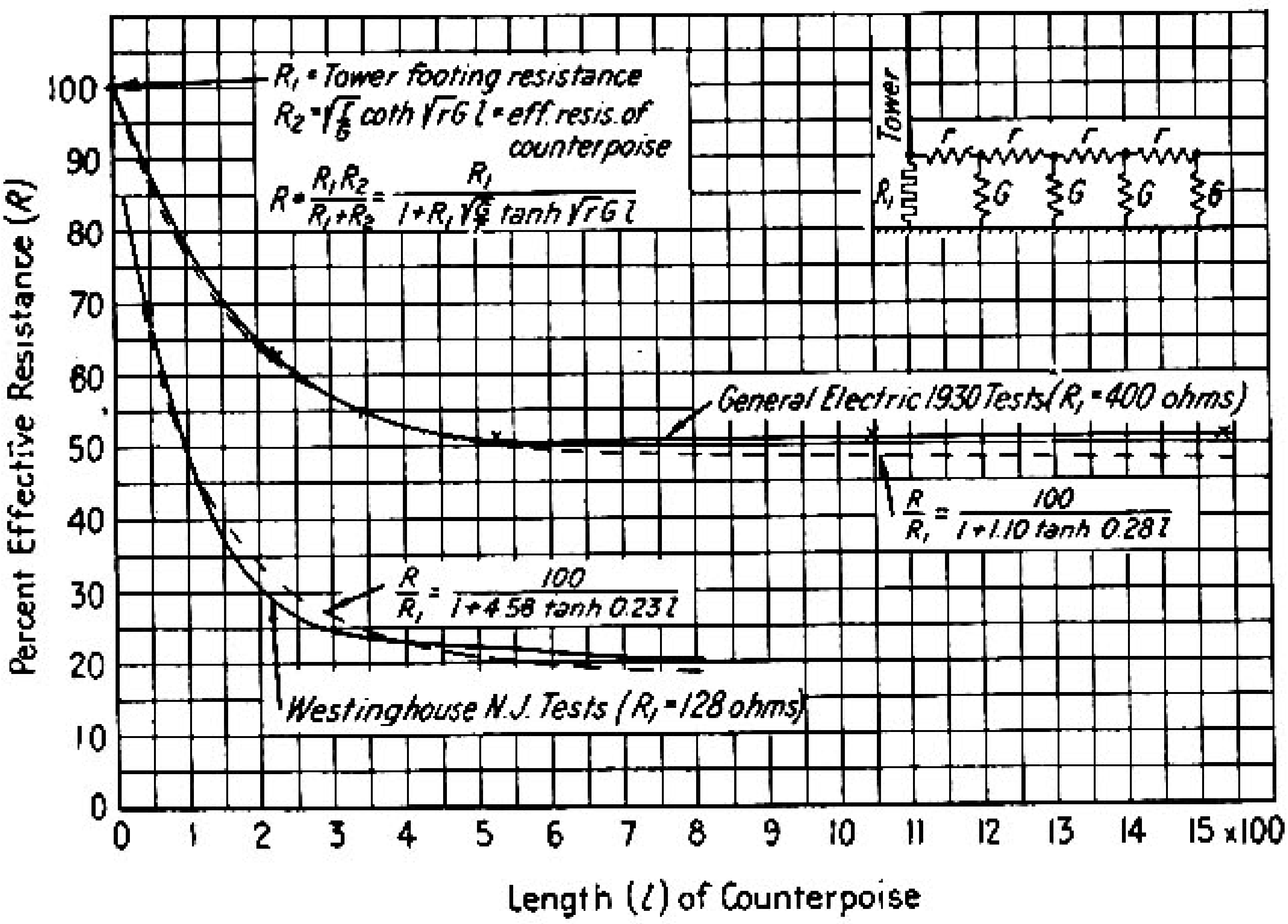


FIG. 86.—Effect of a Counterpoise

b. Estimate the maximum induced voltages, based on Figs. 71 and 72.

c. If the induced voltage is too high for the line insulation, calculate the protective ratio of the ground wires by Equation (272). The maximum induced voltage is then

$$e = \alpha\, G\, h\ (\text{P.R.})$$

and the number and arrangement of the ground wires should be such as to reduce this value below the flashover voltage of the insulators.

d. Estimate the probability of the line being struck by lightning, from curves such as indicated in Fig. 77 and the data resulting from (*a*).

e. Determine the maximum permissible tower footing resistance.

$$R = 1.25 \frac{\text{(insulator flashover in time 2 } T)}{\text{(assumed current in tower)}}$$

where T = *length of span in microseconds*. The insulator flashover is to be taken on a rising front. To prevent most flashovers the tower current may be taken as 100,000 amperes, but for practically complete immunity from flashover it should be taken as 300,000 amperes.

f. From Fig. 85 determine the necessary clearance of the ground wire above the nearest line conductor at midspan.

g. Check the clearance found in (*f*) against that necessary for shielding effect, as given in Fig. 78.

h. Install extra ground wires on the first few towers away from the station or terminal equipment, and reduce the tower footing resistances on these towers to the lowest possible values.

SUMMARY OF CHAPTER X

Ground wires will reduce by approximately 50 per cent the voltage appearing on a transmission line by electrostatic induction. The exact reduction factor is called the *protective ratio* and depends only on the number and arrangement of the ground wires and the height above ground of the line wire for which computed. It is independent of the presence of the other line wires. In terms of the surge impedances of the conductors, the protective ratio is given by Equations (272) and (270), in which the ($p_{11} \ldots p_{mm}$) coefficients are replaced by the corresponding surge impedances ($z_{11} \ldots z_{mm}$). The effect of corona can be approximated for in accordance with Equations (273) to (276). Low tower footing resistance is not of such vital importance for induced surges as it is for direct strokes, but even for induced surges it should be kept below 75 ohms.

Ground wires used for protection against direct strokes should conform to the following requirements:

1. High enough above the line wires to intercept the stroke.
2. High enough above the line wires at midspan to prevent a sideflash during the interval required for the reflections from the tower to return to midspan and relieve the stress.
3. Low enough tower footing resistances to prevent flashover of the insulators. ($V_{\text{Insul.}} \cong 0.8\, R\, I.$)

Ordinarily, condition 2 necessitates a separation of 25 to 35 ft., whereas condition 1 is satisfied by 10 ft. Condition 3 may require tower footing resistances of 10 ohms or less, in which event it may be necessary to use ground rods or counterpoises as discussed in the text. The advantages of lower tower footing resistance are:

(*a*) Lower voltages on the ground and line wires, and across the insulator strings, (*b*) limitation of the region of disturbance to fewer spans, and (*c*) shorter duration of dangerous voltages. It is therefore especially essential to reduce the tower footing resistances on the first few towers away from the station to the lowest possible values. A ground wire which functions properly limits all high voltages due to direct strokes to very short waves, regardless of the time of cloud discharge.

The introduction of extra ground wires for the purpose of reducing the surge impedance is of negligible value.

CHAPTER XI

ARCING GROUNDS AND SWITCHING SURGES

There are two generally accepted theories of arcing grounds, called the *normal-frequency* arc extinction and the *oscillatory-frequency* arc extinction theories respectively, referring to the manner in which the arcs are assumed to go out. By either theory the building up of abnormal voltages is a cumulative recurrent phenomenon. Figs. 87 to 96 inclusive used to illustrate this discussion have been taken from a paper by J. E. Clem.*

Normal-Frequency Arc Extinction—Single-Phase.—Fig. 87 shows a single-phase circuit in which the phase voltages $E_1 = E$ and

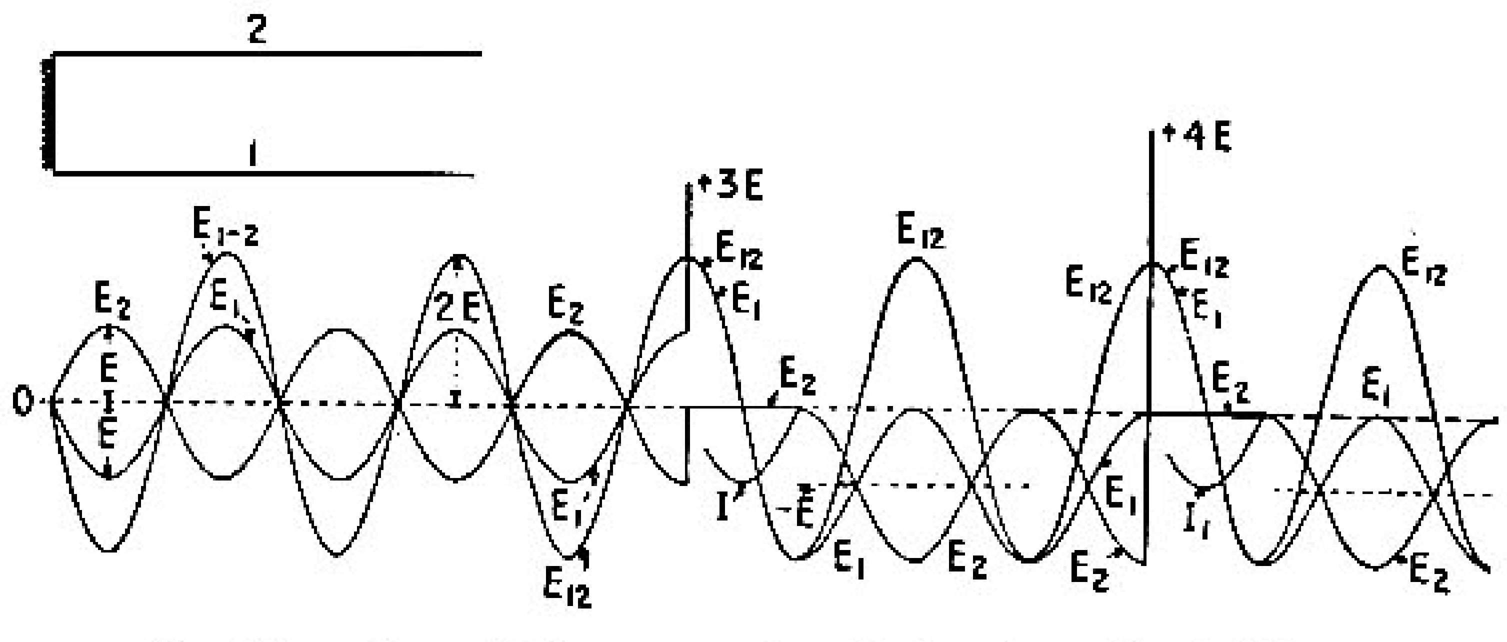


FIG. 87.—Normal Frequency Arc Extinction—Single Phase

$E_2 = E$ are oscillating at normal frequency about the neutral voltage $E_n = 0$. There is thus a voltage difference between lines of $E_{12} = 2\,E$. If, now, line 2 arcs to ground as it reaches its maximum negative value $-E$, then line 1 must go to $E_1 = 2\,E$, but since the circuit contains both inductance and capacitance (to ground and between lines) the potential E_1 will oscillate about the final value $2\,E$ with an amplitude of

$$A_1 = (\text{final voltage}) - (\text{initial voltage})$$

$$= \quad (+2\,E) \quad - \quad (+E) = E \tag{299}$$

* "Arcing Grounds and Effect of Neutral Grounding Impedance," *A.I.E.E. Trans.*, Vol. 49.

The maximum voltage reached by the oscillations is

$$\begin{aligned} V_1 &= \text{(final voltage)} + \text{(amplitude of oscillation)} \\ &= \quad (2\,E) \quad + \quad (E) = 3\,E \end{aligned} \tag{300}$$

The high-frequency oscillation rapidly dies out, and when the normal-frequency arc current (which is practically 90° lagging) passes through zero the arc goes out. At this instant $E_1 = -2\,E$ and $E_2 = 0$, so that

$$\begin{aligned} Q_1 &= K_{11}\,e_1 + K_{12}\,e_2 = -\,K_{11}\,2\,E + 0 \\ Q_2 &= K_{12}\,e_1 + K_{22}\,e_2 = -\,K_{12}\,2\,E + 0 \\ \hline Q_1 + Q_2 &= -\,2\,E\,(K_{11} + K_{12}) \end{aligned}$$

When the arc extinguishes, these charges diffuse over the two lines and establish (through an oscillation) the average potential P (which is also the neutral voltage E_n)

$$P = \frac{(Q_1 + Q_2)}{K_{11} + 2\,K_{12} + K_{22}} = \frac{-2\,(K_{11} + K_{12})\,E}{K_{11} + 2\,K_{12} + K_{22}}$$

If the two lines are at the same elevation, then $K_{22} = K_{11}$ and $P = -\,E$, and the normal-frequency oscillation takes place about P as an axis until E_2 again arcs to ground. If this second arc occurs when $E_2 = -\,2\,E$, then as before

$$\left.\begin{aligned} A_1 &= (+2\,E) - (0) \quad = 2\,E \\ V_1 &= (2\,E) \quad + (2\,E) = 4\,E \end{aligned}\right\}$$

The high-frequency oscillation dies out, and again, as the current is passing through zero, that is, when $E_1 = -\,2\,E$, the arc goes out and the sequence repeats, the conditions being identical with those which obtained when the arc went out the first time. Thus $4\,E$ is the highest voltage obtainable on the basis of normal-frequency arc extinction on a single-phase line.

Normal-Frequency Arc Extinction—Three-Phase.—The analysis for a three-phase system is similar to that for the single-phase system described above. Referring to Fig. 88, suppose that line 2 arcs to ground when $E_2 = -\,E$ and $E_1 = E_3 = E/2$. Then since $E_{12} = E_{23} = 3\,E/2$ at this instant

$$\left.\begin{aligned} A_1 &= (3\,E/2) - (E/2) = E \\ V_1 &= (3\,E/2) + (E) \quad = 5\,E/2 \end{aligned}\right\}$$

The arc goes out when $I_1 + I_3 = 0$, that is, when $E_1 = -3E/2$, $E_2 = 0$, $E_3 = -3E/2$, and the charges on the lines are

$$Q_1 = K_{11}\,e_1 + K_{12}\,e_2 + K_{13}\,e_3 = -(K_{11} + K_{13})\,3E/2$$
$$Q_2 = K_{21}\,e_1 + K_{22}\,e_2 + K_{23}\,e_3 = -(K_{21} + K_{23})\,3E/2$$
$$Q_3 = K_{31}\,e_1 + K_{32}\,e_2 + K_{33}\,e_3 = -(K_{31} + K_{33})\,3E/2$$

$$Q_1 + Q_2 + Q_3 = -(K_{11} + K_{12} + K_{23} + 2\,K_{13} + K_{33})\,3E/2$$

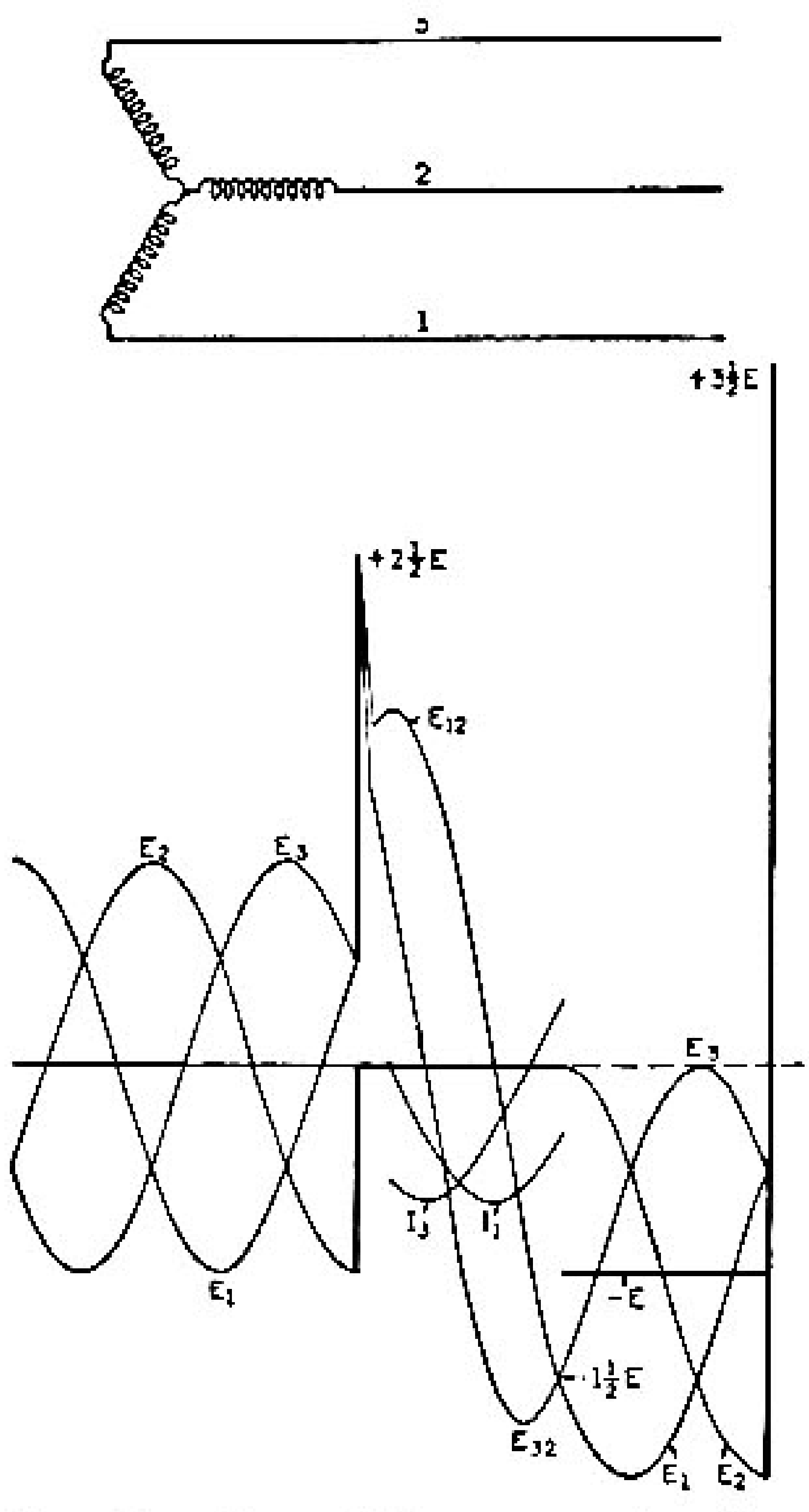


FIG. 88.—Normal Frequency Arc Extinction—Three-Phase

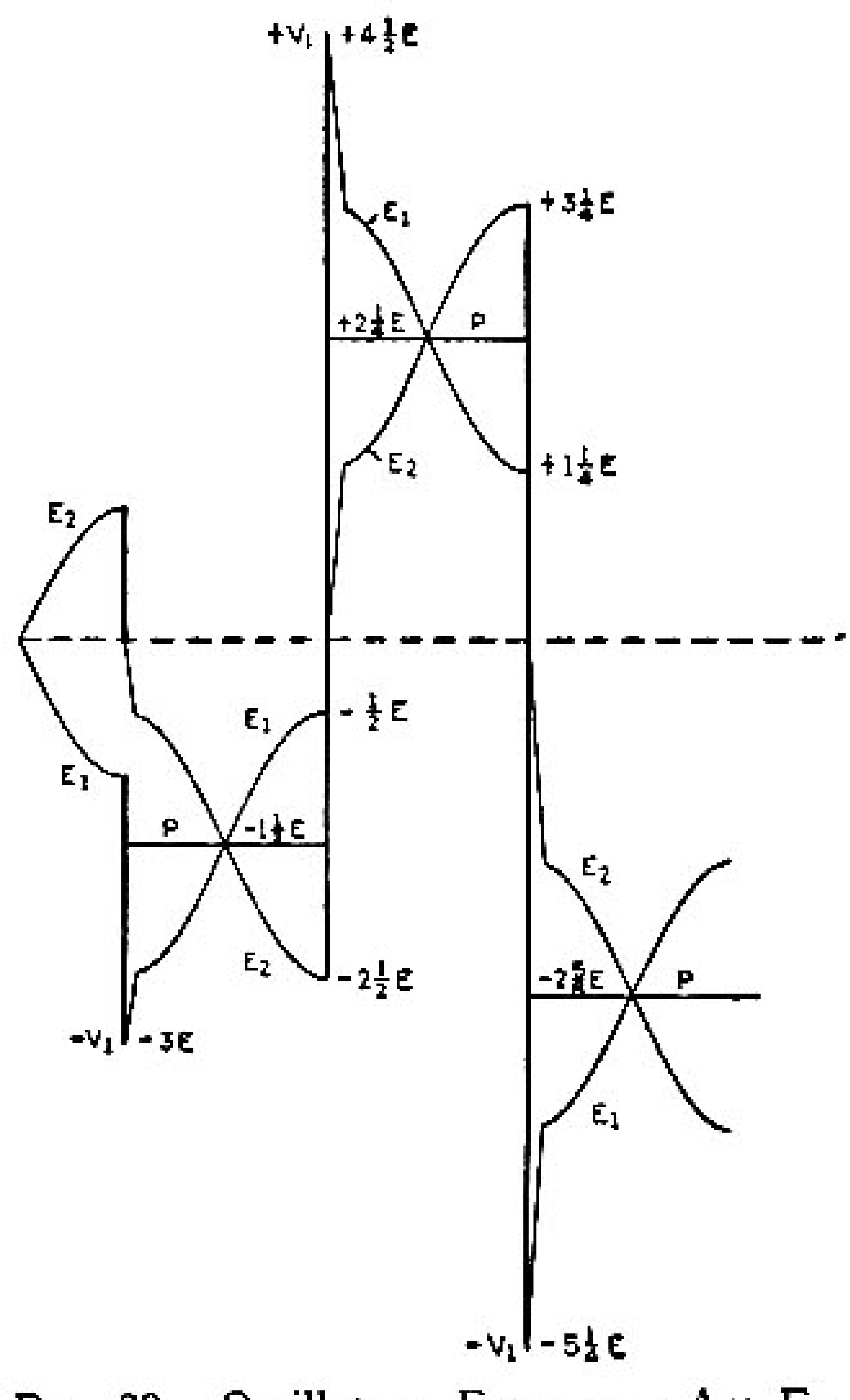


FIG. 89.—Oscillatory Frequency Arc Extinction—Single-Phase

When the arc goes out these charges diffuse through the system and establish (through an oscillation) an average potential (which is also the potential of the neutral E_n)

$$P = \frac{Q_1 + Q_2 + Q_3}{(K_{11} + K_{22} + K_{33} + 2\,K_{12} + 2\,K_{13} + 2\,K_{23})}$$
$$= \frac{-3\,(K_{11} + K_{33} + K_{12} + K_{23} + 2\,K_{31})\,E}{2\,(K_{11} + K_{22} + K_{33} + 2\,K_{12} + 2\,K_{23} + 2\,K_{31})}$$

If the line is completely transposed so that $K_{11} = K_{22} = K_{33}$ and $K_{12} = K_{23} = K_{31}$ this gives $P = -E$ and the phase voltages oscillate at normal frequency and amplitude about P as an axis until the arc restrikes, when $E_2 = -2E$, $E_1 = E_3' = -E/2$. Then

$$\left.\begin{aligned} A &= (1.5\,E) - (-0.5\,E) = 2\,E \\ V &= (1.5\,E) + (2\,E) \qquad = 3.5\,E \end{aligned}\right\}$$

the arc goes out again at $E_1 = E_3 = -1.5\,E$. Thereafter the sequence may repeat, but with no further increase in voltage.

Oscillatory-Frequency Arc Extinction—Single-Phase.—Referring to Fig. 89, suppose that line 2 arcs to ground at $E_2 = E$ and $E_1 = -E$. Then since $E_{12} = -2\,E$ the amplitude of the resulting high-frequency oscillation is

$$\left.\begin{aligned} A_1 &= (-2\,E) - (-E) = -E \\ V_1 &= (-2\,E) + (-E) = -3\,E \end{aligned}\right\}$$

The arc is assumed to go out as the high-frequency current is passing through zero, that is when $E_1 = V_1$. The charges diffuse through the system and establish the average potential

$$P = \frac{(K_{11} + K_{12})(-3\,E)}{K_{11} + 2\,K_{12} + K_{22}} \cong -\frac{3}{2}E$$

and the normal-frequency oscillation is about P as an axis. At $E_2 = -2.5\,E$ and $E_1 = -0.5\,E$ the faulty line E_2 again arcs to ground, and

$$\left.\begin{aligned} A &= (2\,E) - (-0.5\,E) = 2.5\,E \\ V &= (2\,E) + (2.5\,E) \quad = 4.5\,E \end{aligned}\right\}$$

The arc goes out at $E_1 = V$ and the charges diffuse through the system to establish an average potential

$$P \cong \frac{4.5\,E}{2} = 2.25\,E$$

The sequence of events then repeats, so that at the kth arc

$$\begin{aligned} P_{(k-1)} &= \frac{V_{(k-1)}}{2} \\ E_1 &= P_{(k-1)} - E \quad \text{before arcover} \\ A_k &= (-2\,E) - (P_{(k-1)} - E) = -E - P_{(k-1)} \\ V_k &= (-2\,E) + (-E - P_{(k-1)}) = -3\,E - P_{(k-1)} \\ &= -3\,E - \frac{V_{(k-1)}}{2} \end{aligned}$$

Substituting repeatedly into the right-hand member until $k = 2$ and summing the resulting series, there results

$$
\begin{aligned}
-V_k &= 3E + \frac{V_{(k-1)}}{2} \\
&= 3E + \frac{1}{2}\left[3E + \frac{V_{(k-2)}}{2}\right] \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&= 3E + \frac{3E}{2} + \frac{3E}{4} + \ldots + \frac{3E}{2^{(k-2)}} + \frac{V_1}{2^{(k-1)}} \\
&= 3E\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{k-1}}\right] \rightarrow 6E \text{ as } k \rightarrow \infty
\end{aligned}
$$

which is the maximum voltage reached.

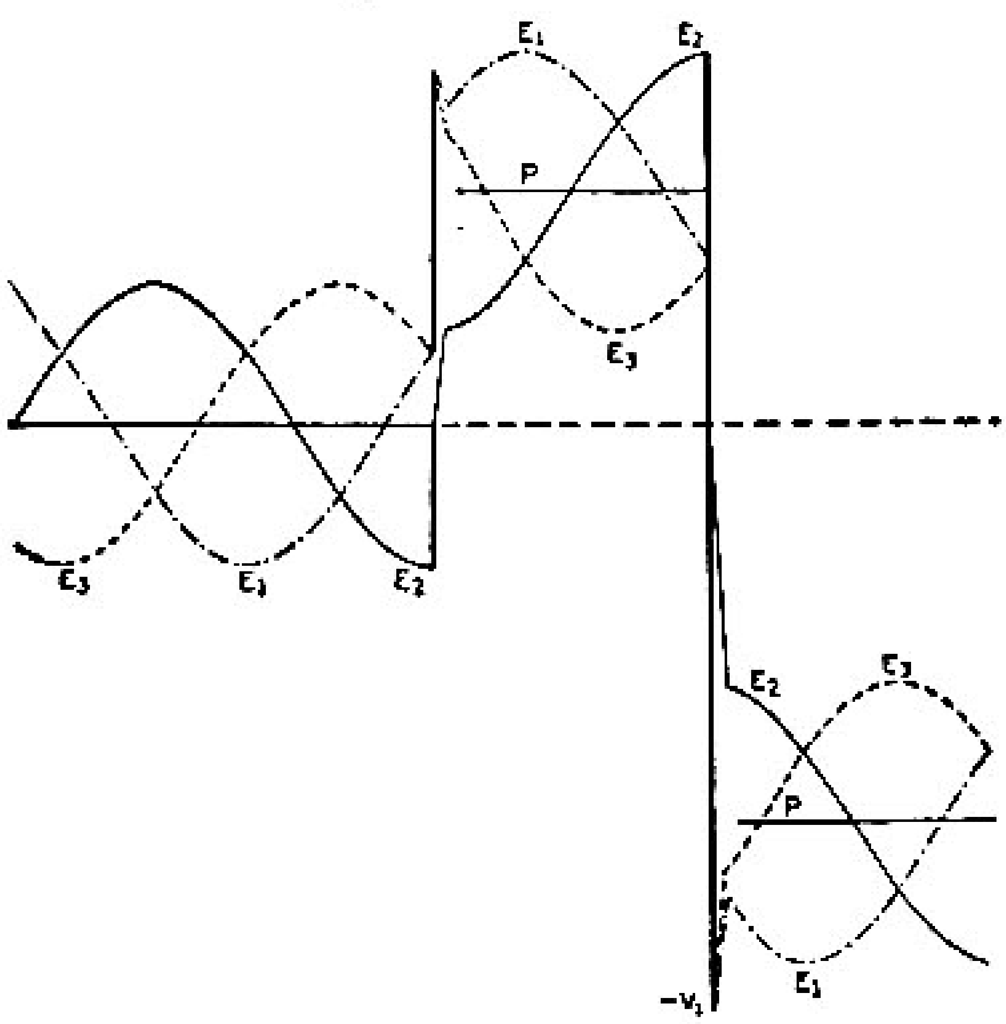


FIG. 90.—Oscillatory Frequency Arc Extinction—Three-Phase

Oscillatory-Frequency Arc Extinction — Three-Phase — Isolated Neutral.—Referring to Fig. 90 and ignoring all damping or neutral connection, there is

$$
\begin{aligned}
&P_{(k-1)} \cong \frac{2}{3} V_{(k-1)} \\
&\quad E_1 = E_3 = P_{(k-1)} - 0.5\,E \quad \text{before arcover} \\
&\quad A_k = (-1.5\,E) - (P_{(k-1)} - 0.5\,E) = -E - P_{(k-1)} \\
&\quad V_k = (-1.5\,E) + (-E - P_{(k-1)}) = -2.5\,E - P_{(k-1)} \\
&\qquad = -2.5\,E - \frac{2}{3} V_{(k-1)}
\end{aligned}
$$

The maximum voltage is realized when $V_{(k-1)} = -V_k$. Therefore $1 - 2/3\ V_{max} = -2.5\ E$, or $V_{max} = 7.5\ E$.

ARCING GROUND TRANSIENTS—THREE-PHASE—NEUTRAL IMPEDANCE

The previous discussion is based on many simplifying assumptions, such as ignoring the characteristics of the transients and all losses and decrements. The present section inquires more into the nature of the transients. Referring to Fig. 91, let Z_n be the neutral impedance and

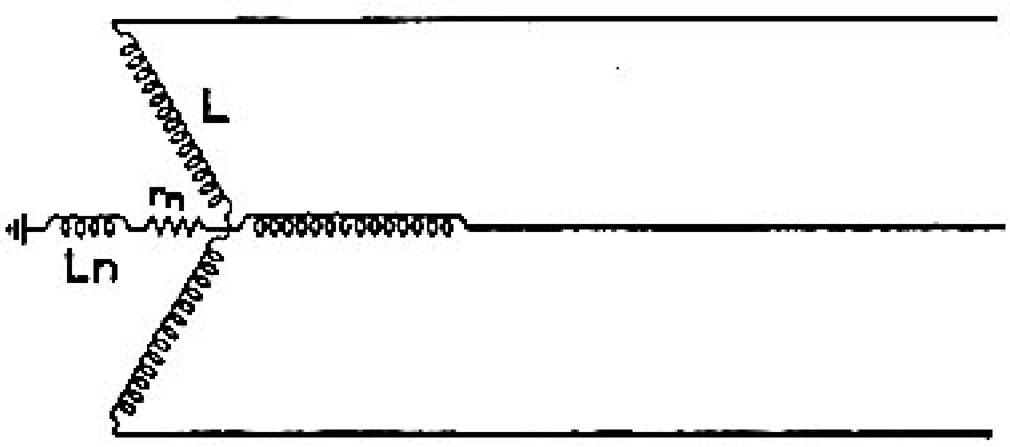


FIG. 91.—Neutral Impedance

Z the series impedance per phase, including the line and connected equipment. Then the phase voltages are

$$\left.\begin{aligned} e_1 &= e_n + E_1 - Zi_1 \\ e_2 &= e_n + E_2 - Zi_2 \\ e_3 &= e_n + E_3 - Zi_3 \end{aligned}\right\} \tag{301}$$

where

e_n = neutral voltage.

$E_1, E_2, E_3 = E \cos (\lambda t + \theta_1),\ E \cos (\lambda t + \theta_2),\ E \cos (\lambda t + \theta_3)$

= generated voltages $(E_1 + E_2 + E_3 = 0)$.

i_1, i_2, i_3 = phase currents.

The neutral voltage is

$$e_n = -Z_n i_n = -Z_n (i_1 + i_2 + i_3) \tag{302}$$

The currents flowing into the capacitances are

$$\left.\begin{aligned} p\,Q_1 &= p\,(K_{11}\,e_1 + K_{12}\,e_2 + K_{13}\,e_3) \\ p\,Q_2 &= p\,(K_{21}\,e_1 + K_{22}\,e_2 + K_{23}\,e_3) \\ p\,Q_3 &= p\,(K_{31}\,e_1 + K_{32}\,e_2 + K_{33}\,e_3) \end{aligned}\right\} \tag{303}$$

Substituting (303) and (302) into (301), and rearranging, there result the differential equations of the system, *for no line grounded and no connected load.* These equations further ignore the distributed nature of the line constants.

$$\left.\begin{aligned}
E_1 &= [1 + pZK_{11} + pZ_n(K_{11} + K_{21} + K_{31})]e_1 \\
&\qquad + [pZK_{12} + pZ_n(K_{12} + K_{22} + K_{32})]e_2 \\
&\qquad + [pZK_{13} + pZ_n(K_{13} + K_{23} + K_{33})]e_3 \\
E_2 &= [pZK_{21} + pZ_n(K_{11} + K_{21} + K_{31})]e_1 \\
&\qquad + [1 + pZK_{22} + pZ_n(K_{12} + K_{22} + K_{32})]e_2 \\
&\qquad + [pZK_{23} + pZ_n(K_{13} + K_{23} + K_{33})]e_3 \\
E_3 &= [pZK_{31} + pZ_n(K_{11} + K_{21} + K_{31})]e_1 \\
&\qquad + [pZK_{32} + pZ_n(K_{12} + K_{22} + K_{32})]e_2 \\
&\qquad + [1 + pZK_{33} + pZ_n(K_{13} + K_{23} + K_{33})]e_3
\end{aligned}\right\} \quad (304)$$

For a completely transposed line, $K_{11} = K_{22} = K_{33} = K$ and $K_{12} = K_{13} = K_{23} = K'$. Let

$$\psi(p) = 1 + pZK + pZ_n(K + 2K')$$

$$\phi(p) = pZK' + pZ_n(K + 2K')$$

Then (304) becomes

$$\left.\begin{aligned}
E_1 &= \psi e_1 + \phi e_2 + \phi e_3 \\
E_2 &= \phi e_1 + \psi e_2 + \phi e_3 \\
E_3 &= \phi e_1 + \phi e_2 + \psi e_3
\end{aligned}\right\} \quad (305)$$

from which there are the differential equations

$$\left.\begin{aligned}
&(\psi - \phi)(\psi + 2\phi)e_1 \\
&\qquad = (\psi + \phi)E_1 - \phi(E_2 + E_3) = (\psi + 2\phi)E_1 \\
&(\psi - \phi)(\psi + 2\phi)e_2 \\
&\qquad = (\psi + \phi)E_2 - \phi(E_1 + E_3) = (\psi + 2\phi)E_2 \\
&(\psi - \phi)(\psi + 2\phi)e_3 \\
&\qquad = (\psi + \phi)E_3 - \phi(E_1 + E_2) = (\psi + 2\phi)E_3
\end{aligned}\right\} \quad (306)$$

By (303) and (302)

$$i_n = i_1 + i_2 + i_3 = (K + 2K')p(e_1 + e_2 + e_3) \quad (307)$$

$$e_n = -Z_n i_n = -(K + 2K')Z_n p(e_1 + e_2 + e_3) \quad (308)$$

Let $Z = (r + p\,L)$ and $Z_n = (r_n + p\,L_n)$. Then

$$\psi = 1 + [r\,K + r_n\,(K + 2\,K')]\,p + [K\,L + (K + 2\,K')\,L_n]\,p^2$$

$$\phi = \qquad [r\,K' + r_n\,(K + 2\,K')]\,p + [K'L + (K + 2\,K')\,L_n]p^2$$

$$(\psi - \phi)\,(\psi + 2\,\phi) = \frac{1}{\Omega_0^2\,\omega_0^2}\,(p^2 + 2\,\beta\,p + \Omega_0^2)(p^2 + 2\,\alpha\,p + \omega_0^2) \quad (309)$$

where

$$2\,\alpha = (r + 3\,r_n)/(L + 3\,L_n), \qquad 2\beta = r/L$$

$$\omega_0^2 = 1/(L + 3\,L_n)\,(K + 2\,K'), \qquad \Omega_0^2 = 1/L\,(K - K')$$

$$\omega = \sqrt{\omega_0^2 - \alpha^2}, \qquad \Omega = \sqrt{\Omega_0^2 - \beta^2}$$

Hereby the solution for e_1, e_2, or e_3 in (306) is

$$\begin{aligned} e = {} & \epsilon^{-\alpha t}\,(A\,\cos\omega\,t + B\sin\omega\,t) \\ & + \epsilon^{-\beta t}\,(C\cos\Omega\,t + D\sin\Omega\,t) \\ & + \frac{\Omega_0^2\,E\,[2\,\beta\,\lambda\sin(\lambda\,t + \theta) + (\beta^2 - \lambda^2 + \Omega^2)\cos(\lambda\,t + \theta)]}{[(\beta^2 + \lambda^2 + \Omega^2)^2 - 4\,\lambda^2\,\Omega^2]} \end{aligned} \quad (310)$$

where $E\cos(\lambda\,t + \theta)$ is the generated phase voltage. The integration constants A, B, C, D are, in general, different for e_1, e_2, and e_3, and must be redetermined for each change in circuit conditions, that is, each time that the arc clears or reignites. The phase voltages thus consist of a double frequency transient superimposed upon a steady-state term. One component of the transient is primarily controlled by the neutral impedance and the capacitance to ground, whereas the other component is entirely independent of the neutral impedance. As far as the steady-state term of (310) is affected it is sufficient to take $\beta \cong 0$ and Ω large compared with λ, whereupon the steady-state term reduces to $E\cos(\lambda\,t + \theta)$, and

$$\left.\begin{aligned} e_1 \cong {} & \epsilon^{-\alpha t}\,(A_1\cos\omega\,t + B_1\sin\omega\,t) \\ & + \epsilon^{-\beta t}\,(C_1\cos\Omega\,t + D_1\sin\Omega\,t) + E\cos(\lambda\,t + \theta_1) \\ e_2 \cong {} & \epsilon^{-\alpha t}\,(A_2\cos\omega\,t + B_2\sin\omega\,t) \\ & + \epsilon^{-\beta t}\,(C_2\cos\Omega\,t + D_2\sin\Omega\,t) + E\cos(\lambda\,t + \theta_2) \\ e_3 \cong {} & \epsilon^{-\alpha t}\,(A_3\cos\omega\,t + B_3\sin\omega\,t) \\ & + \epsilon^{-\beta t}\,(C_3\cos\Omega\,t + D_3\sin\Omega\,t) + E\cos(\lambda\,t + \theta_3) \end{aligned}\right\} \quad (311)$$

in which θ_1, θ_2, and θ_3 are at 120° intervals.

By (305)

$$\left.\begin{aligned}(\psi - \phi)(e_1 - e_2) &= (E_1 - E_2)\\(\psi - \phi)(e_2 - e_3) &= (E_2 - E_3)\\(\psi - \phi)(e_3 - e_1) &= (E_3 - E_1)\end{aligned}\right\} \qquad (312)$$

Consequently the differences of the line voltages are quite independent of the neutral impedance, since $(\psi - \phi)$ does not involve the neutral impedance constants. Therefore, in (311), since the differences must not contain α and ω,

$$\left.\begin{aligned}A_1 = A_2 = A_3 = A\\B_1 = B_2 = B_3 = B\end{aligned}\right\}$$

and, comparing (311) with (301), the common term is the neutral voltage (if no line is grounded)

$$e_n = \epsilon^{-\alpha t}(A \cos \omega t + B \sin \omega t) \qquad (313)$$

and the other transient terms are the transient impedance drops in the phases. There then remain in (311) eight integration constants which may be determined from the initial conditions of the four voltages (e_1, e_2, e_3, e_n) and the four currents (i_1, i_2, i_3, i_n).

Isolated Neutral.—If the neutral is isolated, putting $L_n = \infty$ in the above equations, there is $\alpha = 0$ and $\omega = 0$, so that the phase voltages take the form

$$e = A + \epsilon^{-\beta t}(C \cos \Omega t + D \sin \Omega t) + E \cos (\lambda t + \theta) \qquad (314)$$

As soon as the transient term is dissipated the *average* potentials are those determined by the total charge on the system at the initial instant, since no charge can leak off with an isolated neutral. Thus if at the beginning of the transient $e_1 = e_3 = V$ and $e_2 = 0$ there is by (303)

$$Q_1 + Q_2 + Q_3 = 2\,(K + 2\,K')\,V \qquad (315)$$

At the end of the transient all three lines are at the same average potential E_n (by virtue of a common neutral), and (303) then gives

$$Q_1 + Q_2 + Q_3 = 3\,(K + 2\,K')\,E_n \qquad (316)$$

and therefore, comparing (314), (315), and (316)

$$A = E_n = \frac{2}{3}\,V \qquad (317)$$

But at $t = 0$, (314) (taking $\theta_2 = 0$) gives

$$\left.\begin{aligned} e_1 &= V = A + C_1 + E\cos\theta_1 = 2\,V/3 + C_1 - E/2 \\ e_2 &= 0 = A + C_2 + E\cos\theta_2 = 2\,V/3 + C_2 + E \\ e_3 &= V = A + C_3 + E\cos\theta_3 = 2\,V/3 + C_3 - E/2 \end{aligned}\right\} \tag{318}$$

From which

$$\left.\begin{aligned} C_1 &= V/3 + E/2 \\ -\,C_2 &= 2\,V/3 + E \\ C_3 &= V/3 + E/2 \end{aligned}\right\} \tag{319}$$

Substituting the three equations of type (314) in (303) and putting $i_1 = i_2 = i_3 = 0$ at $t = 0$ yields the equations for the determination of the D constants. If $\beta/\Omega \cong 0$, the D constants vanish entirely, and the phase voltages then are

$$\left.\begin{aligned} e_1 &= \frac{2}{3}\,V + \left(\frac{E}{2} + \frac{1}{3}\,V\right)\epsilon^{-\beta t}\cos\Omega\,t + E\cos(\lambda\,t + 120°) \\ e_2 &= \frac{2}{3}\,V - \left(E + \frac{2}{3}\,V\right)\epsilon^{-\beta t}\cos\Omega\,t + E\cos(\lambda\,t) \\ e_3 &= \frac{2}{3}\,V + \left(\frac{E}{2} + \frac{1}{3}\,V\right)\epsilon^{-\beta t}\cos\Omega\,t + E\cos(\lambda\,t - 120°) \end{aligned}\right\} \tag{320}$$

It will be observed, if β is not excessive, that the maximum voltage occurs on the *faulty* line, rather than on the good lines. If the arc does not strike again for a half cycle or more of normal frequency, the high-frequency oscillation (Ω) of (320) will have vanished by that time.

One Line Grounded.—The previous equations apply under the condition that no line is grounded. If one line, say No. 2, is grounded, then $e_2 = 0$ in (301) and (303). The currents in the good lines are $i_1 = p\,Q_1$ and $i_3 = p\,Q_3$ respectively, but the current i_2 of the faulty line is defined by the second equation of (301). Substituting (303) in (301) and (302), the simultaneous equations connecting the four unknowns are:

$$\left.\begin{aligned} e_1 &= e_n + E_1 - Z\,p\,(K\,e_1 + K'\,e_3) \\ 0 &= e_n + E_2 - Z\,i_2 \\ e_3 &= e_n + E_3 - Z\,p\,(K'\,e_1 + K\,e_3) \\ e_n &= -\,Z_n\,p\,(K + K')\,(e_1 + e_3) - Z_n\,i_2 \end{aligned}\right\} \tag{321}$$

Solving for the neutral voltage there is

$$e_n = \frac{-Z_n E \cos(\lambda t + \theta_2)}{(Z + Z_n) + (Z + 3 Z_n) Z p (K + K')} \tag{322}$$

Regardless of Z_n, this differential equation in symbolic notation is at least a cubic, and therefore the roots must be found by trial. There are three limiting cases, however, for which solutions are readily obtainable.

Case I. Resistances Negligible.—In this case $Z = p L$ and $Z_n = p L_n$, and (322) reduces to

$$e_n = \frac{-L_n E \cos(\lambda t + \theta_2)}{(L + L_n) + (L + 3 L_n) L (K + K') p^2} \tag{322A}$$

and the solution is

$$e_n = A \cos \omega t + B \sin \omega t - \frac{L_n E \cos(\lambda t + \theta_2)}{L (L + 3 L_n)(K + K')(\omega^2 - \lambda^2)} \tag{323}$$

where $\omega^2 = (L + L_n) L (L + 3 L_n)(K + K')$. But λ^2 is small compared to ω^2, and (323) becomes

$$e_n \cong A \cos \omega t + B \sin \omega t - \frac{L_n}{L + L_n} E \cos(\lambda t + \theta_2) \tag{323A}$$

Case II. Line Capacitance Negligible.—In this case $(K + K') = 0$ and (322) reduces to

$$e_n = -\frac{Z_n e \cos(\lambda t + \theta_2)}{Z + Z_n} \tag{322B}$$

and if $Z_n = (r_n + p L_n)$, the solution is

$$e_n = A \epsilon^{-\alpha t} - E\sqrt{\frac{r_n^2 + \lambda^2 L_n^2}{(r + r_n)^2 + \lambda^2 (L + L_n)}} \cos\left[\lambda t - \tan^{-1}\frac{(r_n + \lambda L_n)}{(\alpha r_n - \lambda^2 L_n)}\right] \tag{324}$$

where $\alpha = (r + r_n)(L + L_n)$.

Case III. Z Small Compared with Z_n.—Canceling Z_n from the numerator and denominator, (322) reduces to

$$e_n = \frac{-E \cos(\lambda t + \theta_2)}{1 + 3 (K + K') p (r + p L)} \tag{322C}$$

the solution to which is

$$e_n = \epsilon^{-\alpha t}(A \cos \omega t + B \sin \omega t) - \frac{(\alpha^2 + \omega^2) E [2 \alpha \lambda \sin(\lambda t + \theta_2) + (\alpha^2 + \omega^2 - \lambda^2) \cos(\lambda t + \theta_2)]}{[(\alpha^2 + \omega^2 + \lambda^2)^2 - 4 \omega^2 \lambda^2]} \tag{325}$$

where $\alpha = r/2L$

$$\omega^2 = \left[\frac{1}{3L(K + K')} - \alpha^2\right]$$

Neglecting α and λ in comparison with ω, this becomes

$$e_n \cong \epsilon^{-\alpha t}(A \cos \omega t + B \sin \omega t) - E \cos (\lambda t + \theta_2) \qquad (325A)$$

The foregoing Equations (301) to (325*A*) inclusive show the nature of the transients associated with arcing grounds. From (322) the amount of neutral shift corresponding to a given neutral impedance may be estimated; and from (311) and (313) the amount by which the neutral voltage decays while the arc is out, and the decrement of the high-frequency oscillations, may be determined. These three reduction factors are responsible for a considerable reduction in the maximum voltages due to arcing grounds. J. E. Clem gives the following table, in which the neutral impedance is of such value that the neutral shift on short circuit is not more than two-thirds the normal phase voltage:

MAXIMUM VOLTAGES OF ARCING GROUNDS

	Single-Phase	Three-Phase
Initial arc, isolated neutral	$3E$	$2.5E$
Normal-frequency arc extinction	$4E$	$3.5E$
Oscillatory-frequency arc extinction:		
Isolated neutral, no damping	$6E$	$7.5E$
Isolated neutral, damping	....	$5.3E$–$5.7E$
Resistance in neutral	....	$2.5E$
Reactance in neutral	....	$3.7E$–$4.0E$
Petersen coil in neutral	....	$1.34E$

Extensive tests on three-phase laboratory circuits failed to show a voltage in excess of $3.2E$ on isolated neutral, and the phenomenon seemed to be controlled by normal-frequency arc extinction. There is no rational reason why an arc which initially strikes at normal voltage should require a successively higher voltage for each subsequent arc; and yet the arcing ground theories described above demand such a sequence in order that the voltages may be built up cumulatively to the maximum values given in the table. It is much more likely that just the reverse is true, that is, that the subsequent arcs will require less voltage to ignite them. It is probable, therefore, that arcing ground voltages do not reach $4E$, even on isolated neutral systems.

PETERSEN COIL

If x_1, x_2, and x_0 are the positive, negative, and zero sequence reactances respectively of a system, as viewed from the point of fault, then according to the theory of symmetrical components the fault current for a one-line-to-ground fault may be found by connecting $(x_2 + x_0)$ to the positive sequence diagram at the point of fault, Fig. 92. Obvi-

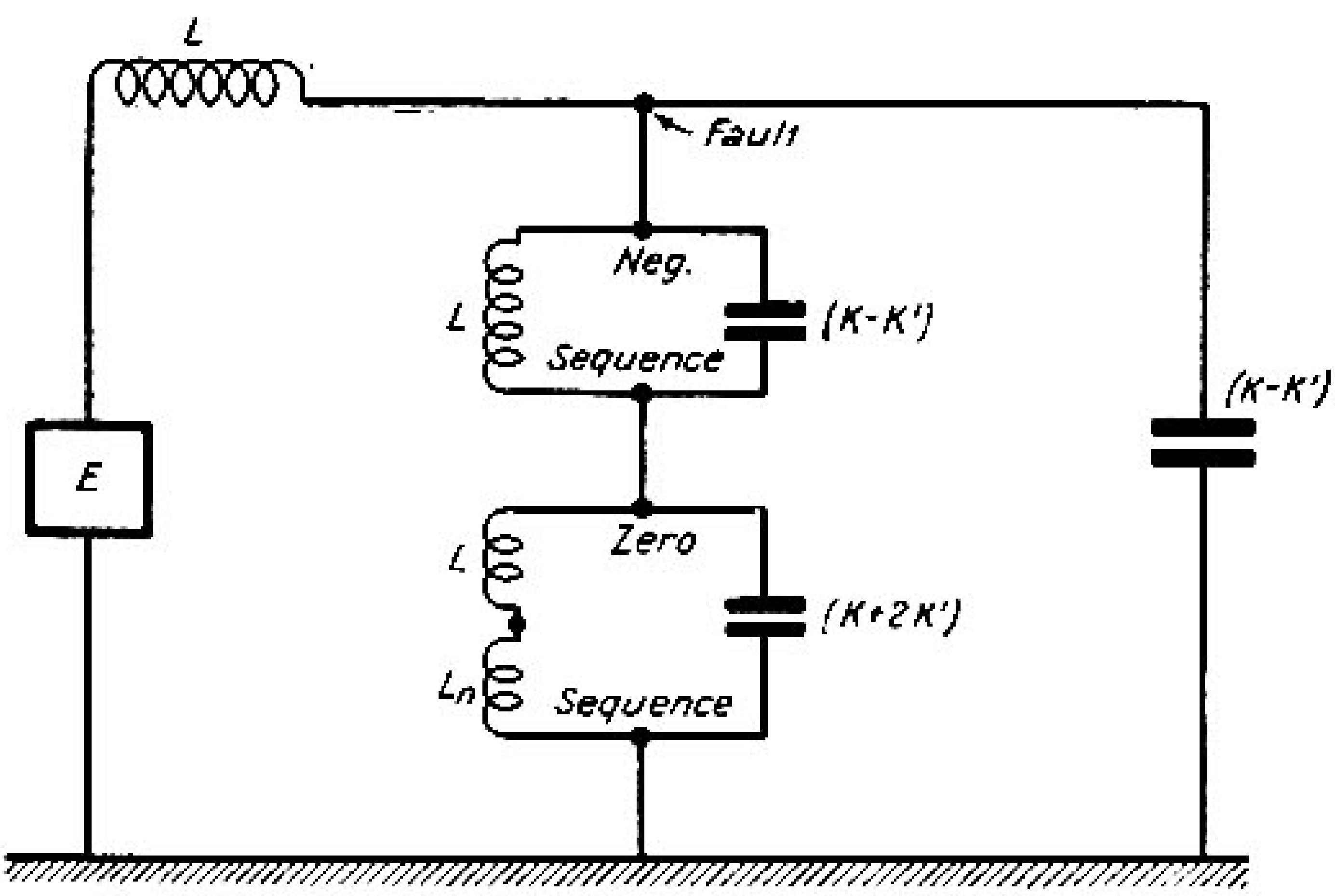


FIG. 92.—Fault Conditions with a Petersen Coil

ously, then, no current can flow in the fault, and consequently a normal-frequency arc can not be supported, if

$$x_f = x_0 + x_2 = \infty \tag{326}$$

The effective capacitance of a completely transposed transmission line to positive or negative sequence voltages is defined by

$$\dot{Q}_1 = (K\,\dot{E}_1 + K'\,\dot{E}_2 + K'\,\dot{E}_3) = (K - K')\,\dot{E}_1 \tag{327}$$

and the zero sequence capacitance is defined by

$$\dot{Q}_0 = (K\,\dot{E}_0 + K'\,\dot{E}_0 + K'\,\dot{E}_0) = (K + 2\,K')\,\dot{E}_0 \tag{328}$$

The zero sequence reactance of the circuit shown in Fig. 92, putting $x_c = -1/\lambda\,(K + 2\,K')$, is

$$x_0 = \frac{(x + 3\,x_n)\,(-\,x_c)}{(x + 3\,x_n - x_c)} \tag{329}$$

which is infinite for

$$x_n = \frac{x_c - x}{3} \tag{330}$$

A neutral reactance of this value is called a Petersen coil. Such a coil will cause the arc to extinguish by preventing the flow of normal frequency follow current.

SWITCHING SURGES

The scope of this book does not contemplate a detailed discussion of switching surges. It will suffice to review briefly the subject (practically verbatim) along the lines covered by Park and Skeats,* and to point out the principal characteristics of the phenomena. In a general way, switching surges are similar to arcing ground transients, in that the same circuit constants are involved and the initiating cause in both cases is an arc, either to ground or across the circuit-breaker contacts. The study of switching surges and recovery voltages is of primary importance to circuit-breaker engineers, for the interrupting ability of a circuit-breaker depends upon its capacity to increase the dielectric strength across its contacts at a faster rate than the rate at which the voltage is built up.

There are three major aspects to the problem of switching surges:

1. Normal-frequency effects of recovery voltages.
2. High-frequency effects of recovery voltages.
3. Interruption of charging currents and the building up of abnormal voltages by successive reflections.

Normal-Frequency Effects.—The normal-frequency voltage of the first phase to clear of a short circuit depends upon the type of fault, decrement factor of the excitation, displacement of the machine rotor, the direct and quadrature machine reactances, and other circuit constants. Park and Skeats give for the *maximum* voltage of the first phase to clear

$$e_m = k_g\, k_d\, k_q\, E \tag{331}$$

where

k_g depends upon the ground conditions.
k_d = decrement or " change in excitation " factor = i/i''.
k_q = quadrature reactance factor.
E = normal phase voltage.

* " Circuit Breaker Recovery Voltages," by R. H. Park and W. F. Skeats, *A.I.E.E. Trans.*, Vol. 50, 1931.

Values for these constants are given in the following table:

Type of Fault	k_d	k_q
One-line-to-ground	1.0	1.0
Two-line-to-ground	$\dfrac{2\sqrt{3}\,(s_q''^2 + s_q'' x_0 + x_0^2)^{3/2}}{s_d''(2s_q'' + x_0)^2 + x_0(2s_q'' + x_0)(s_q'' + 2x_0)} \cong 1.73$	1.0
Line-to-line	1.73	1.0
Three-phase s.c.	$\dfrac{3x_0}{s_q'' + 2x_0}$	$\dfrac{s_q''}{s_d''}$
Three-phase ungrounded s.c., or grounded s.c. on ungrounded system	1.5	$\dfrac{s_q''}{s_d''}$

x_q'' = quadrature subtransient reactance of machine.
x_d'' = direct subtransient reactance of machine.
s_q'' = x_q'' + (*external reactance of system*).
s_d'' = x_d'' + (*external reactance of system*).
x_0 = zero sequence reactance of system.
i = short-circuit current at instant of clearing.
i'' = initial inrush of current.

Since $i'' \geq i$ it is sufficient to take $k_d = 1.0$.

For a full discussion of the effects of the type of fault, rotating machine characteristics, amortisseur winding, displacement, initial load current, and saturation, reference should be made to the original paper. The discussion is avoided here, since a complete understanding of it leads to an involved study of synchronous machine theory. Very briefly:

Effect of	
Amortisseur winding	Reduces both the quadrature reactance factor k_q and the decrement factor k_d, and thereby considerably reduces the recovery voltage and the duty on the circuit-breaker.
Displacement	The existence of a d-c. component in the s-c. current may cause the recovery voltage wave to start near zero instead of the crest, thus reducing the instantaneous value of the recovery voltage.
Initial load current	The current i is the sum of the load and fault currents, and therefore k_d is increased.
Saturation	Reduces the quadrature reactance factor k_q and therefore reduces the recovery voltage.

High-Frequency Effects.—The presence of capacitance in the windings of transformer and rotating machines, bus structure, bushings, current limiting reactors, and other terminal apparatus, in conjunction with the inductances of windings, provides the necessary facilities for high-frequency oscillations, upon the sudden rise of normal-frequency recovery voltage. Thus because of these oscillations the voltage may overshoot to double values. From the point of view of the circuit-breaker operation, these high-frequency oscillations are of prime importance, because they determine the *rate of rise* of voltage across the contacts, and so fix the ability of the circuit-breaker to rupture the circuit. If there were no capacitance in the circuit the rise of recovery voltage would be abrupt and would immediately cause arcing across the break. Therefore, capacitance is not only an essential element in the circuit, but the smaller it is the more important does it become, for the rate of rise is the faster. The axis of high-frequency oscillations is the normal-frequency voltage, and the oscillation starts at the arc voltage existing just before interruption. Ordinarily the starting voltage may be taken as zero, but if it is otherwise, the amplitude of oscillation will be increased.

A resistance (of 1000 ohms or less) connected across the terminals of a circuit-breaker materially reduces the amplitude of oscillation and the rate of rise.

One or more transmission lines connected between the breaker and the source of power greatly reduces the rate of rise of the recovery voltage. The following examples, taken from the paper by Park and Skeats, illustrate the method of solution.

A. Circuit 1, Curve 1, Figs. 93 and 94.—Recovery voltage of first phase to clear of a three-phase grounded short circuit on a solidly grounded generator with 3 ohms external reactance.

$x_q'' = 0.77$	$s_q'' = 3.77$	$k_q = 1.05$
$x_d'' = 0.60$	$s_d'' = 3.60$	$k_d = 1.00$
$x_0 = 0.05$	$s_0 = 3.05$	$k_a = 0.93$

The normal-frequency recovery voltage ($E = \sqrt{2}\,8400$) is $e_m = 0.93 \times 1.00 \times 1.05 \times \sqrt{2} \times 8400 = 11{,}600$ volts.

The effective inductance is $L = 3.77/377 = 0.010$.

The bus capacitance is $C = 0.008$ microfarad.

The natural frequency is $f = 1/(2\pi\sqrt{LC}) = 17{,}800$.

The limiting conditions are: $\begin{cases} e = 0 \text{ and } de/dt = 0 & \text{at } t = 0 \\ e = 11{,}600 & \text{at } t = \infty. \end{cases}$

Ignoring the decrements, the recovery voltage is

$$e = 11{,}600\ (1 - \cos 2\,\pi\ 17{,}800\ t)$$

and the maximum rate of recovery voltage is

$$de\ dt = 2\,\pi\, f\, E \sin\ (\pi\ 2) = 1300 \text{ volts per ms.}$$

B. Fig. 95.—Recovery voltage of first phase to clear of a three-phase-to-ground short circuit on an ungrounded generator with 3 ohms external reactances. Circuit constants same as before, but for this type of fault $k_v = 1.50$. Therefore

$$e_m = 1.50 \times 1.00 \times 1.05 \times \sqrt{2} \times 8400 = 18{,}700 \text{ volts.}$$

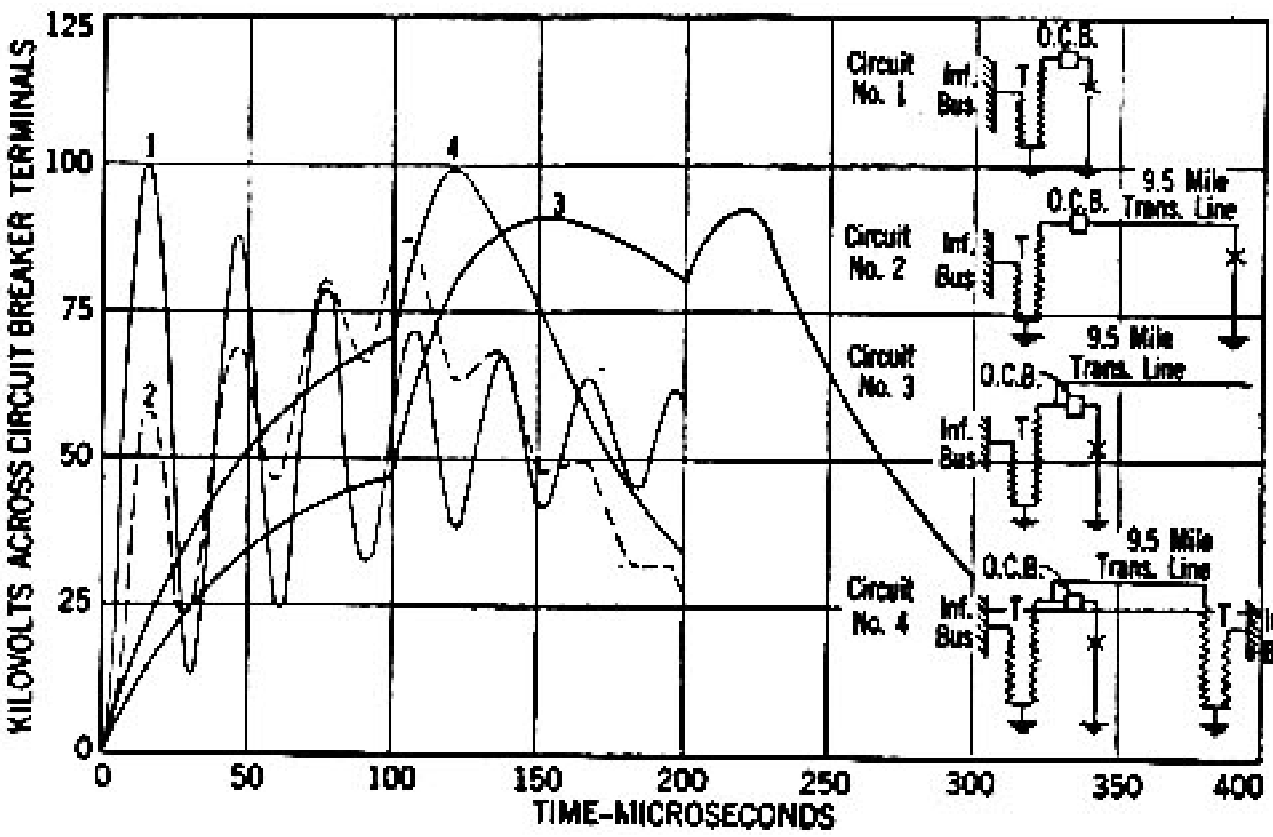


FIG. 93.—Calculated Recovery Voltage Curves Following a Single-Phase Line-to-Ground Short-Circuit at Points Marked X on Circuit Diagrams

System voltage: 66,000. Capacity of transformer in each case: 20,000 kv-a. per phase. Transformer reactance: 10 per cent. Transmission line surge impedance: 400 ohms. Length of transmission line: 9.5 mi.

If the bus capacitance is $C_1 = 0.008$ microfarad to ground (on circuit-breaker side of reactor) and the generator winding has a capacitance to ground of $C_2 = 0.8$ microfarad (assumed to be concentrated at the generator neutral), then the total impedance of the circuit is

$$Z = \frac{1}{p\,C_1} + p\,L + \frac{\dfrac{p\,L}{2}\,\dfrac{1}{p\,C_2}}{\dfrac{p\,L}{2} + \dfrac{1}{p\,C_2}}$$

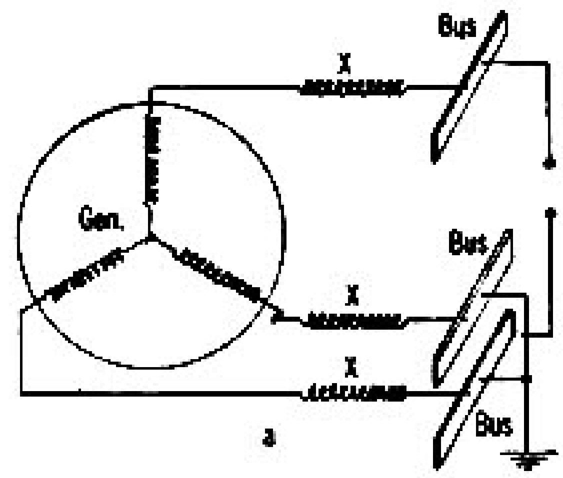
$$= \frac{p^4\,L^2\,C_1\,C_2 + (3\,C_1 + C_2)\,L\,p^2 + 2}{p\,C_1\,(p^2\,L\,C_2 + 2)}$$

and its natural frequencies of oscillation therefore are

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{(3\,C_1 + C_2) + \sqrt{9\,C_1^2 - 2\,C_1\,C_2 + C_2^2}}{2\,L_1\,C_1\,C_2}} = 17{,}950$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{(3\,C_1 + C_2) - \sqrt{9\,C_1^2 - 2\,C_1\,C_2 + C_2^2}}{2\,L_1\,C_1\,C_2}} = 2510$$

The high-frequency oscillation is substantially that of L and C_1 alone, C_2 acting as a short circuit; the low-frequency is that of $L/2$ and C_2 alone, C_1 acting as an open circuit. Thus



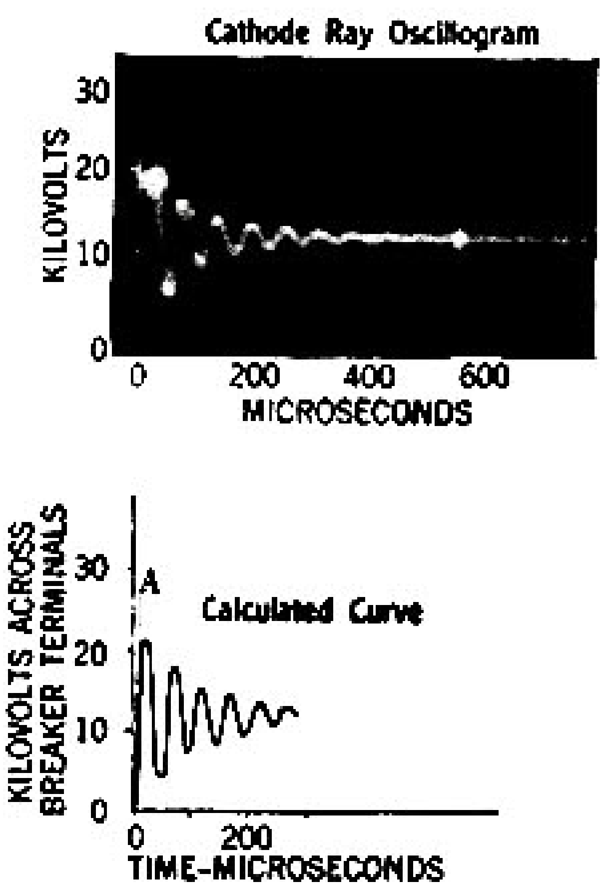


FIG. 94

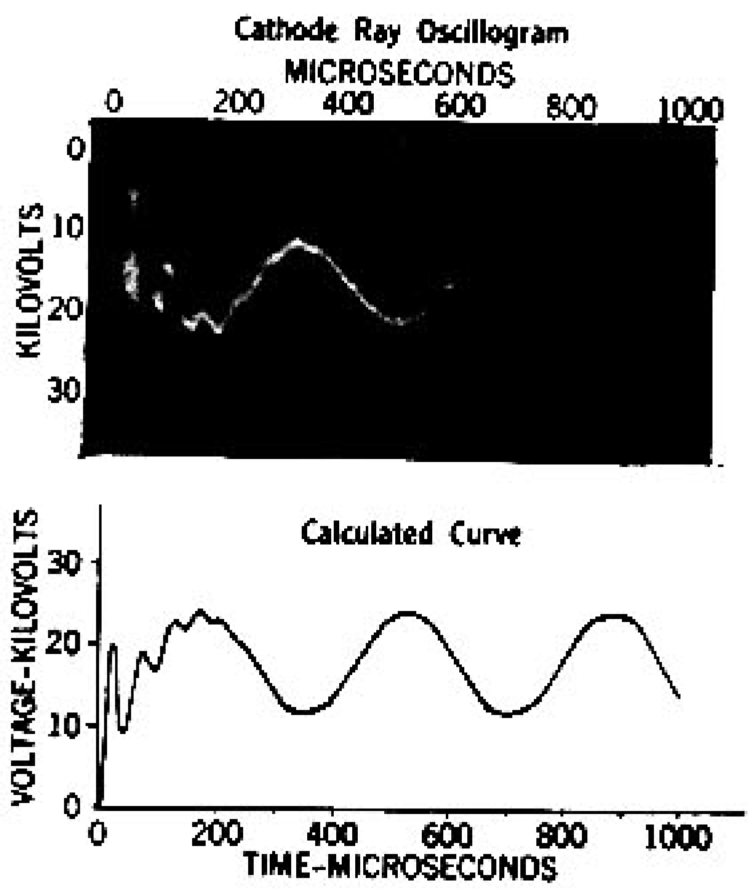


FIG. 95

$$\frac{1}{2\pi}\frac{1}{\sqrt{L\,C_1}} = 17{,}880 \qquad \frac{1}{2\pi}\frac{1}{\sqrt{\frac{L}{2}\,C_2}} = 2510$$

Two-thirds of the final voltage will appear across the open phase, and one-third across the other two phases in parallel. Therefore, the recovery voltage is

$$e = 12{,}470\,(1 - \cos 2\,\pi f_1\,t) + 6230\,(1 - \cos 2\,\pi f_2\,t)$$

C. Circuit 2, Curve 2, Fig. 93.—Recovery voltage of a single-phase-to-ground short circuit 50,000 ft. out on a transmission line; circuit-breaker directly connected to the high side of the transformer; grounded transformer neutral; low-voltage side of transformer connected to an infinite bus.

System voltage	= 38,100 volts line-to-neutral.
Transformer reactance	= 7.25 ohms = 0.0192 henry.
Line reactance	= 7.5 ohms.
Short-circuit current	= 38,100 14.75 = 2580 amperes.
Effective capacitance of transformer	= 0.0012 microfarad.

The rate of change of current at current zero is

$$\frac{di}{dt} = \omega I = 377 \times \sqrt{2} \times 2580 \times 10^{-6} = 1.375 \text{ amperes per ms.}$$

The rate of change of voltage on the line side of the breaker is

$$\frac{de_1}{dt} = Z\,I = 400 \times 1.375 = 550 \text{ volts per ms.}$$

This rate will continue until reflections from the far end of the line return and reverse the polarity.

On the transformer side of the breaker the voltage is

$$e_2 = L\frac{di}{dt}\left(1 - \cos\frac{t}{\sqrt{L\,C}}\right) = 0.0192 \times 1.375\ (1 - \cos 2\pi\ 33{,}000\ t)$$

$$= 26{,}400\ (1 - \cos 2\,\pi\ 33{,}000\ t)$$

The total voltage across the breaker is (t in microseconds)

$$e = e_1 + e_2 = 550\,t + 26{,}400\ (1 - \cos 0.208\ t)$$

D. Circuit 3, Curve 3, Fig. 93.—Single-phase-to-ground short circuit at transmission-line side of circuit-breaker; a second line 50,000 ft. long is connected between the circuit-breaker and the transformer and is open-circuited at the far end; transformer neutral solidly grounded; low side of transformer connected to an infinite bus. Constants same as in previous case.

Short-circuit current = 38,100 7.25 = 5260.

The rate of current at current zero is

$$\frac{di}{dt} = \omega\,I = 377 \times \sqrt{2} \times 5260 \times 10^{-6} = 2.8$$

and therefore, until the reflections return

$$\frac{di}{dt} = \frac{e}{L} + \frac{1}{Z}\frac{de}{dt},$$

the solution to which is

$$e = L\frac{di}{dt}\left(1 - \epsilon^{-(Z/L)t}\right)$$

$$= 53{,}700\left(1 - \epsilon^{-0.0208\,t}\right)$$

At $t = 100$, however, the first reflection arrives, and at 100-ms. intervals thereafter new reflections are superimposed at the transformer terminals. The recovery voltage up to $t = 300$ is plotted in Fig. 93, curve 3.

Interruption of Line Charging Currents.—The process of building up excessive voltages by the interruption of the charging currents of a connected transmission line is as follows. The circuit-breaker contact arc is extinguished when the current is passing through zero and the transmission line completely charged to one polarity $(+E)$. One-half cycle later the transformer terminal voltage has reversed its polarity, but the voltage of the transmission line remains unchanged, so that double leg voltage is across the switch. If this voltage breaks down the gap, a wave $(-2\,E)$ travels down the line and reflects. As the reflected wave reaches the breaker the entire line is charged to a value $(E - 2\,E - 2\,E = -3\,E)$, the current in the arc drops to zero, since the reflected current is equal and opposite to that of the incident wave, and the arc is again interrupted. The transformer side of the breaker then changes polarity during the following half cycle, and if this additional voltage across the contacts is sufficient to reignite the arc, a wave $(4\,E)$ travels down the line, wiping out the previous potential $(-3\,E)$, reflects, and when it again reaches the breaker the line is charged to a value $(-3\,E + 4\,E + 4\,E = 5\,E)$. If the sequence of events repeats indefinitely there is no limit to the voltage which may be built up. Actually, however, the effects of damping limit the voltage to a finite value. Since a half cycle of normal frequency (60 ~) lasts for 8333 ms. there is plenty of time for damping out the traveling waves. Moreover, the process may be halted at any stage depending upon the rate at which the breaker is opening. Suppose that, at a certain instant when the arc is just ready to restrike, the line is charged to a voltage E_0 and the transformer voltage is $a\,E$ where a may have any value between $(+1)$ and (-1). At this instant the dielectric strength of the breaker is $(E_0 - aE)$, and as soon as the arc strikes, a wave $(aE - E_0)$ will travel out on the line, canceling E_0, reflect, and return to the breaker where the arc will extinguish on account of a current zero (the current reflection is equal and opposite to the incident current wave). As the normal frequency reverses polarity, the total voltage across the switch finally is

$$[\alpha\ (2\ aE - 2\ E_0 + E_0) - E] \tag{332}$$

where α is the reduction factor due to losses in that time. This exceeds the previous breakdown voltage by

$$\Delta\ e = (1 - 2\ a\ \alpha + a)\ E - (1 - \alpha)\ E_0 \tag{333}$$

The criterion for final interruption of the arc is that this voltage increment shall be less than the increase $\beta\ E$ of dielectric strength of the switch. Therefore

$$\beta\ E > \Delta\ e = (1 - 2\ a\ \alpha + a)\ E - (1 - \alpha)\ E_0 \tag{334}$$

or

$$E_0 > \left(\frac{1 - 2\ a\ \alpha + a - \beta}{1 - \alpha}\right) E \tag{335}$$

Evidently, then, for a given α and β there is a value of E_0 such that the arc will not be able to restrike. For example, if $a = 1$, $\alpha = 0.8$, $\beta = 0.6$, the arc will not restrike if

$$E_0 > 5\ E$$

Thus the leakage of charge from the line, and the increase of dielectric strength across the circuit-breaker contacts, limit the maximum voltage that can be built up to a definite value. Switching surges as high as 5.5 times normal leg voltage have been recorded on transmission lines in operation. The building up of excessive voltages by the above process can be prevented by using a high-resistance leak to drain off the line charge or by using a breaker designed to increase its dielectric strength at a sufficient rate.

As a matter of fact, the resulting traveling wave is not rectangular, as assumed above. When the gap is broken down, the distribution of line potential $(-E_0)$ immediately splits up as a pair of traveling waves $(-E_0/2)$, and one of these waves meets the inductance of the transformer at which it builds up a voltage $(-E_0\epsilon^{-\tau})$, where $\tau = t\ Z/L$. The other wave reflects at the open end and continues to feed the inductances for a time equal to four times the length of the line. While this is going on, another wave is being impressed upon the transmission line by the discharge of the normal-frequency recovery voltage through the inductance of the transformer, and this is $E(1 - \epsilon^{-\tau})$. The resultant voltage of the line therefore is

$$e = -E_0\epsilon^{-\tau} + E(1 - \epsilon^{-\tau}) = (E_0 + E)(1 - \epsilon^{-\tau}) - E_0 \qquad (336)$$

and the current is

$$i = \frac{E_0 + E}{Z}(1 - \epsilon^{-\tau}) \qquad (337)$$

Calling $E_0 + E = E'$, the *changes* in voltage and current after each reflection (counting time anew at each reflection) are:

$$\left.\begin{aligned} e &= E'(1 - \epsilon^{-\tau}) \\ i &= \frac{E'}{Z}(1 - \epsilon^{-\tau}) \end{aligned}\right\} \text{Until first reflection arrives.} \qquad (338)$$

$$\left.\begin{aligned} e_1 &= E'\,2\,\tau\,\epsilon^{-\tau} \\ i_1 &= \frac{E'}{Z}\,2\,(1 - \epsilon^{-\tau} - \tau\,\epsilon^{-\tau}) \end{aligned}\right\} \text{Increase due to first reflection.} \qquad (339)$$

$$\left.\begin{aligned} e_2 &= E'\,2\,\tau\,\epsilon^{-\tau}(1 - \tau) \\ i_2 &= E'\,2\,(-1 + \epsilon^{-\tau} + \tau\,\epsilon^{-\tau} - \tau^2\,\epsilon^{-\tau}) \end{aligned}\right\} \text{Increase due to second reflection.} \qquad (340)$$

The differences between the potential of the line at any instant, and its potential just before breakdown of the gap, are plotted as curves V_1, V_2, and V_3, and the gap currents as curves A_1, A_2, and A_3 in Fig. 96. Curves V_1 and A_1 apply to a line whose length is such that a wave can travel twice its length in a time equal to the time constant of the circuit (L/Z). Curves V_2, A_2 apply to a line twice as long, and curves V_3, A_3 to a line four times as long.

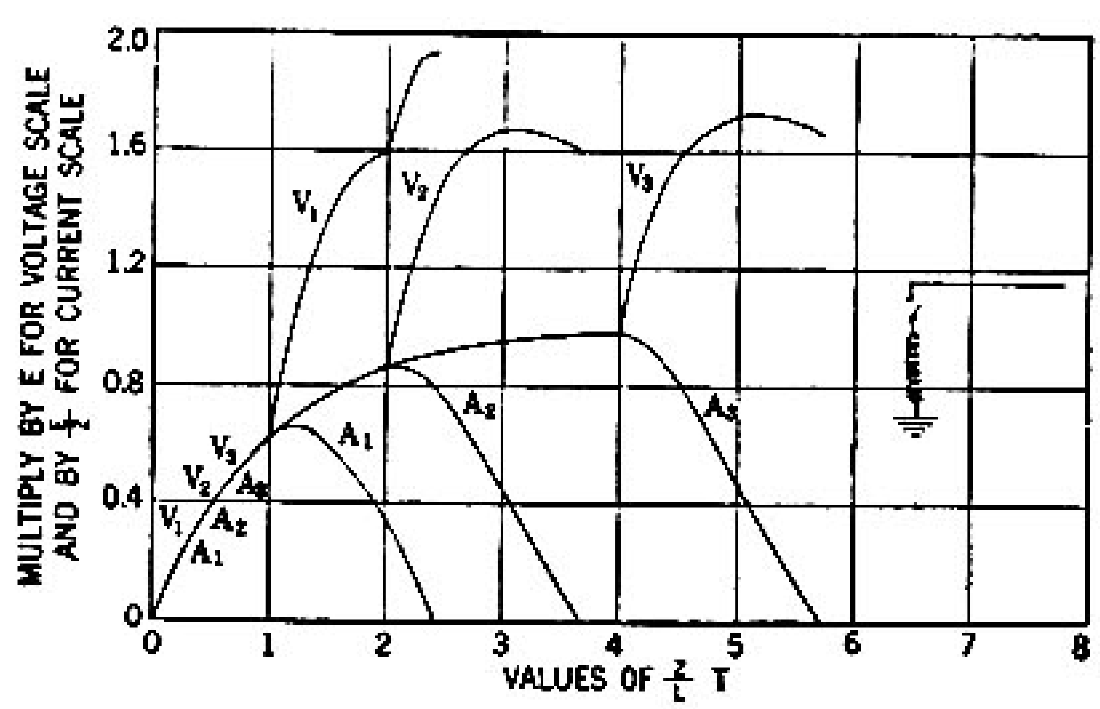


FIG. 96.—Current and Voltage Surges upon the Sudden Connection of a Transmission Line to the High Side of a Transformer when no Other Line is Connected There

SUMMARY OF CHAPTER XI

There are two generally recognized theories of arcing grounds, called the *normal-frequency* and *oscillatory-frequency* arc extinction theories respectively, pertaining to the manner in which the arcs are assumed to go out. By either of these theories the building up of abnormal voltages is a cumulative process involving successively increased residual charges on the good lines, and transient high-frequency oscillations about the potentials established by these residual charges as axes of oscillations. The bound charges depend upon the capacitance coefficients of the circuit, whether single- or three-phase, and the leakage through the neutral impedance. The nature of the high-frequency oscillation is governed by the capacitance and inductance coefficients of the line, the neutral impedance, and the grounding of the faulty line; Equations (311), (323), (324), and (325). The oscillatory-frequency arc extinction theory yields the maximum voltages due to arcing grounds. The possible values given by this theory are tabulated in the text. On a grounded neutral system, abnormal arcing ground voltages are impossible. On an isolated neutral system, voltages as high as 5.7 times normal are theoretically possible, but can be prevented by the use of Petersen coils.

Switching surges may be conveniently considered in two parts—the low-frequency and high-frequency effects, respectively. When a load or fault is interrupted by a circuit-breaker operation there follows a comparatively slow electromagnetic transient in the connected rotating machines, which transient depends upon the type of fault, the machine characteristic constants, and the excitation system. The calculation of the recovery voltage of the first phase to clear thus rests on an intimate understanding of synchronous machine theory, but fortunately reduces to a quite simple formula, Equation (331), whose application does not depend upon much more than a routine procedure. Upon the sudden rise of the normal-frequency recovery voltage there ensues a high-frequency oscillation by virtue of the inductance of windings and lines, and the capacitance of windings, lines, bushings, etc. It is this high-frequency oscillation which establishes the rate of rise of voltage across the circuit-breaker contacts, and therefore determines the ability of the breaker to interrupt the circuit. There are also involved, as components of the high-frequency oscillations, the repeated reflections of traveling waves on the connected transmission lines, initiated by the circuit-breaker action in rupturing the circuit. These repeated reflections, in conjunction with successive reignitions of the arc between circuit-breaker contacts, are responsible for the cumulative building up of excessive voltages due to switching. However, there are on record cathode-ray oscillograms of switching surges of several times normal voltage which do not exhibit the characteristics called for by the theory discussed in this chapter. It appears, therefore, that there is still a fertile field for research concerning switching surges, particularly with reference to the possibility of realizing initial voltage jumps of the order of five times normal voltage.

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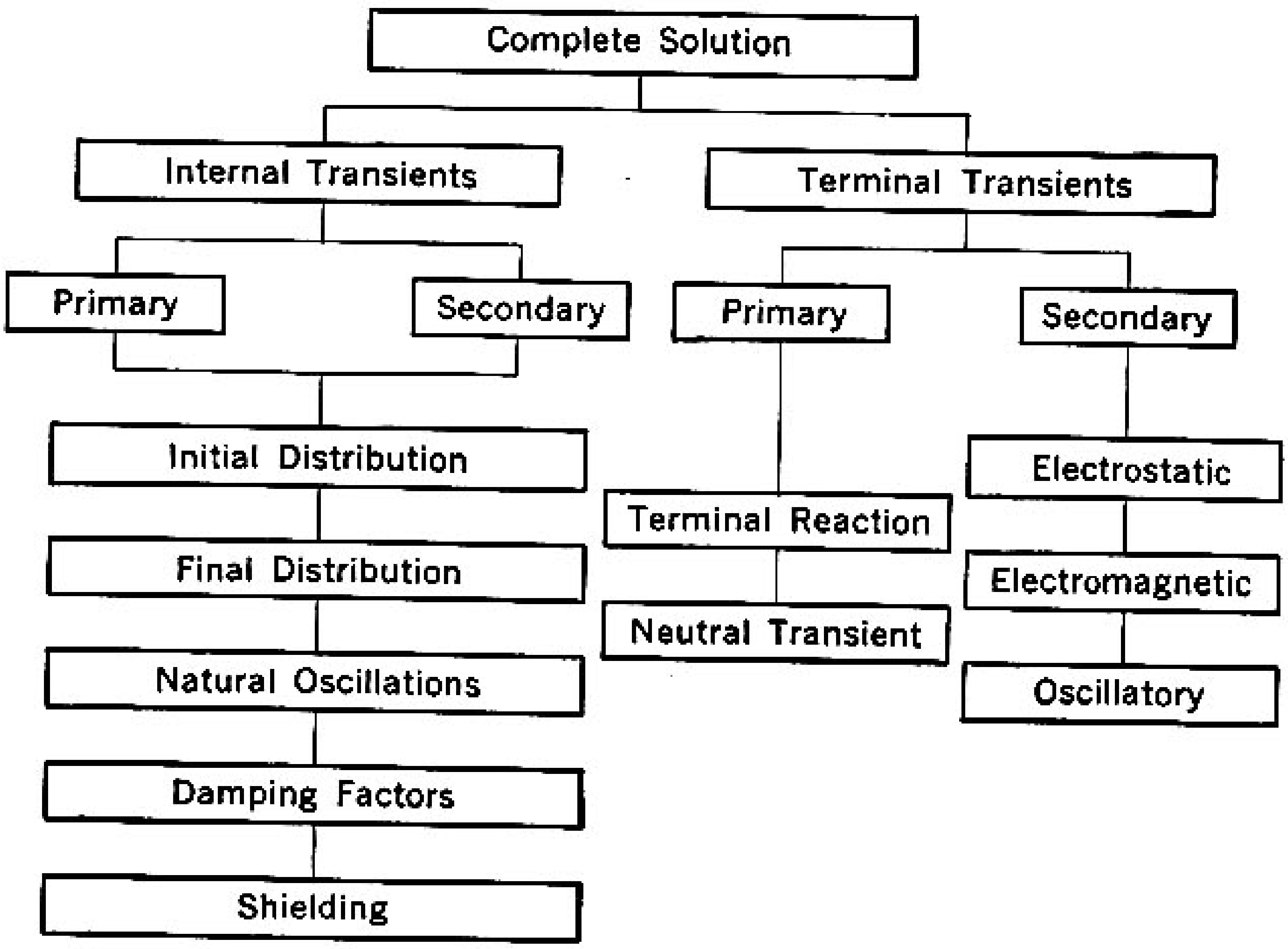
Part II

HIGH-FREQUENCY OSCILLATIONS AND TERMINAL TRANSIENTS OF TRANSFORMERS

INTRODUCTION TO PART II

GENERAL CLASSIFICATION OF TRANSFORMER HIGH-FREQUENCY TRANSIENTS

The remaining chapters of this book are devoted to the analysis of the internal oscillations and terminal transients of transformers, or other distributed windings, subjected to the impact of traveling waves. For convenience, the study of these high-frequency transients is considered according to the following scheme:



In this arrangement, items on or near the same level are closely associated in the analysis. Thus the *initial distribution*, *primary terminal reaction* (*initially*), and *electrostatic component* of the secondary terminal transient are all determined by the same equations. Likewise the *final distribution*, *neutral transient*, *primary terminal reaction* (*later stages*), and the *electromagnetic component* of the secondary terminal transient are all determined by the same equations. The *oscillatory component* of the secondary terminal transient as well as that of the *neutral transient* develop from the *natural oscillations* of the circuit. Therefore, since the *primary terminal reaction* and the *neutral transients* are each a compromise between two effects, they have been placed on lines between these other items. It is thus seen that the foregoing chart provides a most expressive classification of the several aspects of the problem.

A rigorous solution of the complete circuit, including neutral impedances, secondary connections, and the distributed circuit constants of the winding, is out of the question. The mathematical difficulties very soon become insurmountable, and even could general solutions be obtained, they would no doubt be too complicated to be of much use from an engineering point of view. Recourse is therefore had to approximate equivalent circuits, the range of whose validity can be established quite definitely from theoretical considerations, and verified experimentally. Briefly, the solutions for the internal transients of a two-winding transformer with grounded or isolated terminals show that:

1. The primary internal oscillations can be calculated quite accurately by ignoring the secondary entirely and using an " effective inductance " in the equations.
2. While the secondary terminal transient consists of three terms, only the electromagnetic component is of practical importance (if the terminals are not open), and this component has a very simple equivalent circuit, Chapter XV.
3. The primary terminal transients (at line and neutral) may be calculated by a very simple equivalent circuit, Chapter XV.

Chapters XII and XIII dealing with the internal transients are rather involved, mathematically. But the general procedure is quite simple, and is carried out in the following steps: *

a. The differential equations of the circuit are derived.
b. The terminal conditions are specified.
c. The initial distributions at the instant of impact of an infinite rectangular traveling wave are determined.
d. The final distributions (or axes of oscillations) are determined.
e. The complete solution to satisfy the above is obtained, making use of the general identity:

Initial distribution = *final distribution* +
transient terms at first instant (I)

* This procedure was outlined in K. W. Wagner's 1915 paper, Reference 2 of the Bibliography for Part II.

In order to express the two sides of this equation in common terms, Fourier expansions of the initial and final distributions are made.

f. The solutions for other than infinite rectangular waves are obtained through Duhamel's theorem.

g. If terminal impedances are involved, or waves are applied simultaneously at two or more terminals, the principle of superposition is used. The appropriate equivalent circuit for terminal impedances permits the voltage at each terminal impedance to be calculated. Then considering each as an applied voltage, with the other terminal impedances short-circuited, the internal transient due thereto may be calculated, and then

Complete solution = Σ *solutions for each terminal voltage with other terminal impedances short-circuited* (II)

h. The necessary and sufficient conditions which must obtain if oscillations are to be prevented are next investigated. A necessary condition, but not always a sufficient one, from (I) is

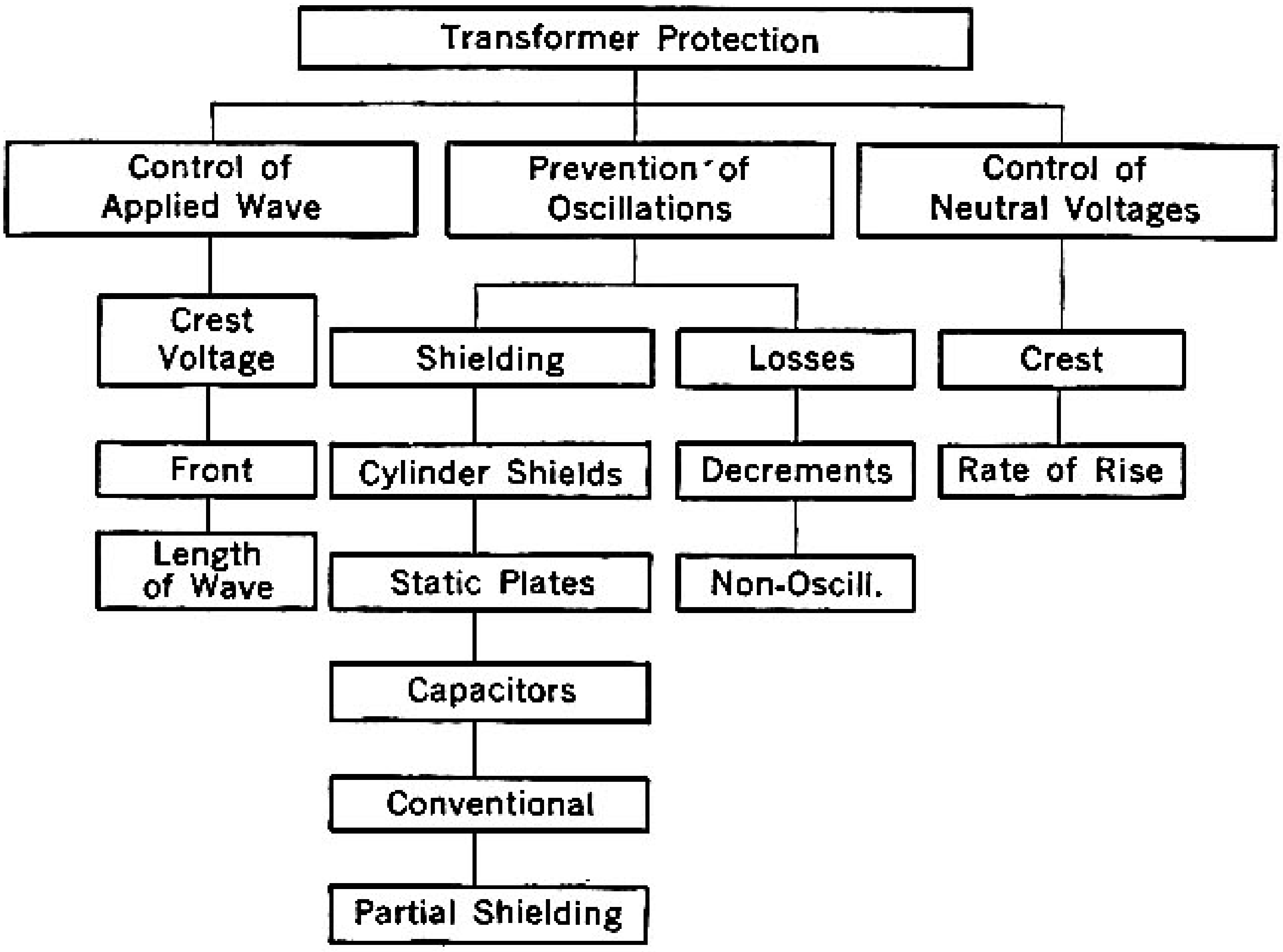
Initial distribution = *final distribution* (III)

The transient oscillations in transformer windings are responsible for excessive voltages to ground, from turn-to-turn and between coils. These excessive voltages and gradients are mitigated by the suppression of internal oscillations, the control of the applied wave, and the control of the neutral voltage—which comprise the three major objectives of all schemes for transformer protection. Protective plans may be classified according to the chart on page 210.

The crest voltage and length of the applied wave are controlled by the coordinating gap and lightningarrester, as discussed in Chapter V. The wave front, or rate of rise, can be retarded by an inductance in series or a capacitance in shunt, but if the former is used it should be bridged by a suitable resistor to prevent it from entering into oscillation with the equivalent capacitance of the transformer. On high-voltage circuits it is not feasible from an economical standpoint to endeavor to retard the wave front by more than a few microseconds.

The crest voltage at the neutral is controlled by Thyrite, and the rate of rise can be retarded by means of capacitors. Detailed calculations are given in Chapter XV.

The idea of increasing the transformer losses by artificial means during the transient has been proposed. If these losses are high



enough, the transient must actually become aperiodic, but otherwise the decrement factors which the losses introduce tend to limit the amplitudes of oscillation. The scheme has been applied to current limiting reactors, Fig. 50.

The various methods of electrostatic shielding which have been considered are discussed in detail in Chapter XIV.

CHAPTER XII

IDEAL TWO-WINDING TRANSFORMER *

The coil assembly of a conventional, concentric-type two-winding transformer is shown in Fig. 97. The high- and low-voltage windings consist of a number of multiple-turn sections wound on supporting insulating cylinders and separated by spacers and oil ducts so as to provide sufficient dielectric strength and to facilitate cooling by oil circulation. The turns and coils of the end sections are provided with extra insulation. Static plates are placed at the ends of the coil

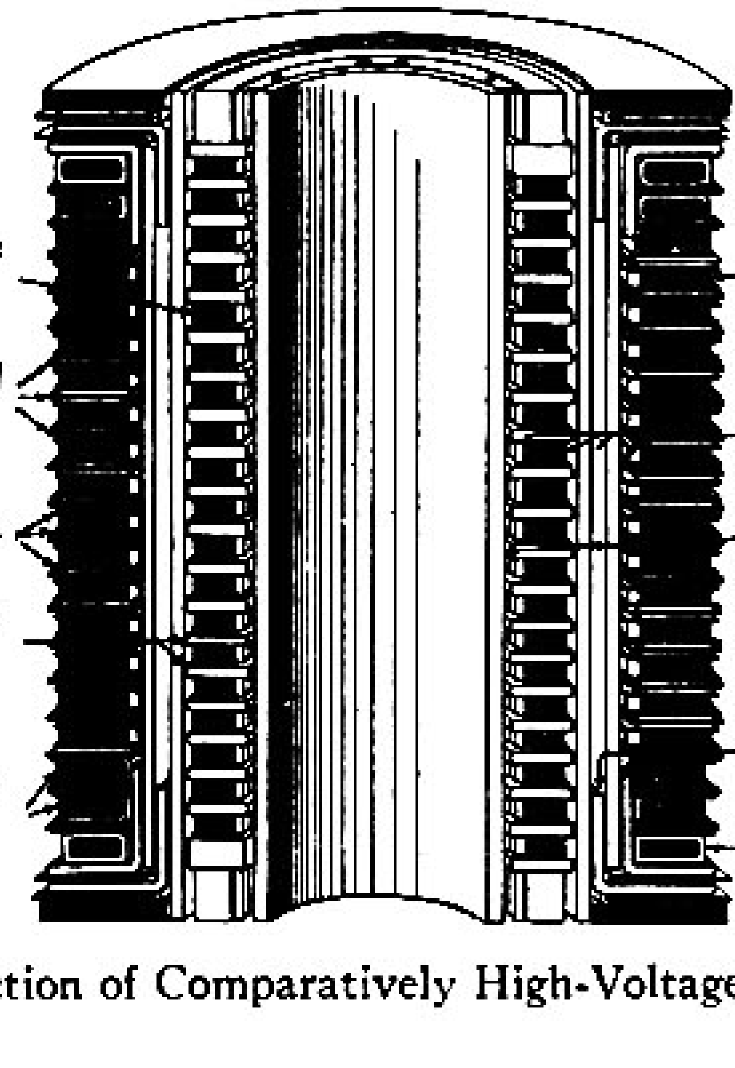


FIG. 97.—Cross-Section of Comparatively High-Voltage Power Transformer

stacks to help equalize the electrostatic distribution at the instant of impact of a traveling wave at the transformer terminals. The insulation between the ends of the stacks and the supporting plates consists of flanged collars and insulating barriers.

Each turn of the winding has a capacitance to all other turns, to the core, and to the tank, so that the complete capacitance network of a transformer is a very complicated circuit. Owing to the close

* L. V. Bewley, "Transient Oscillations of Mutually Coupled Windings," *A.I.E.E. Trans.*, Vol. 51, 1932.

proximity of adjacent sections, however, it is permissible to simplify the circuit to that of Fig. 98, if the investigation is concerned with a turn-to-turn distribution, or even to that of Fig. 99 if only the principal features of transformer internal transients are to be investigated. It will be seen that these simplifications rest upon disregarding the distributed nature of the circuit constants of subsidiary elements of the winding, and neglecting the small capacitances such as from a section to sections beyond its adjacent neighbor. Thus in Fig. 99 the section is adopted as the unit, and the total capacitance of its turns to the adjacent section has been lumped as a series capacitance K_1, its average capacitance to the tank as C_1, and its average capacitance to the low-voltage winding as C_3.

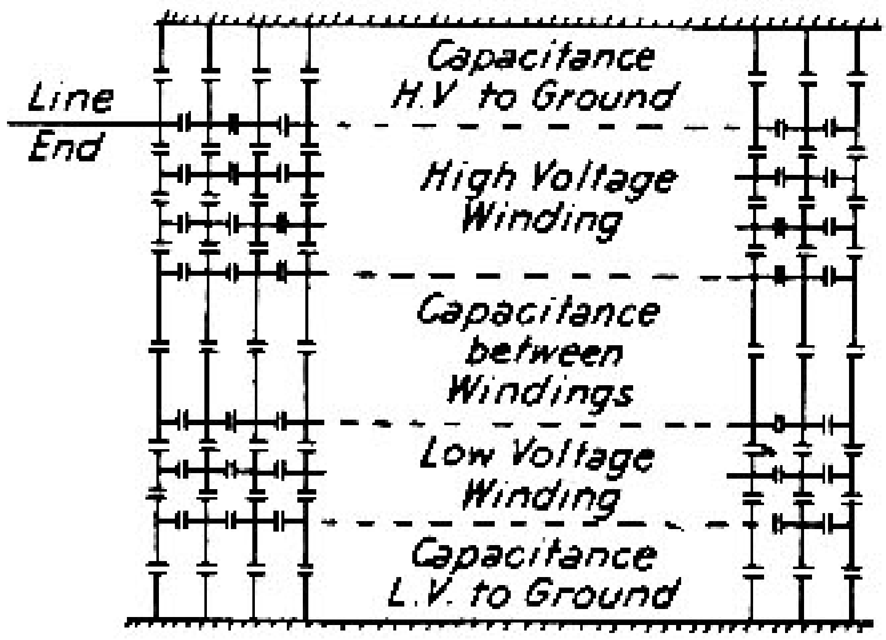


FIG. 98.—Capacitance Network of Transformer Windings

The effective inductance of the windings is made up of three parts: the interlinkages due to flux which is common to all turns of the winding, the partial interlinkages due to flux which is not common to all turns, and the interlinkages due to the other winding. The situation is further complicated by the fact that during the transient the current in different parts of the same winding is different in magnitude and

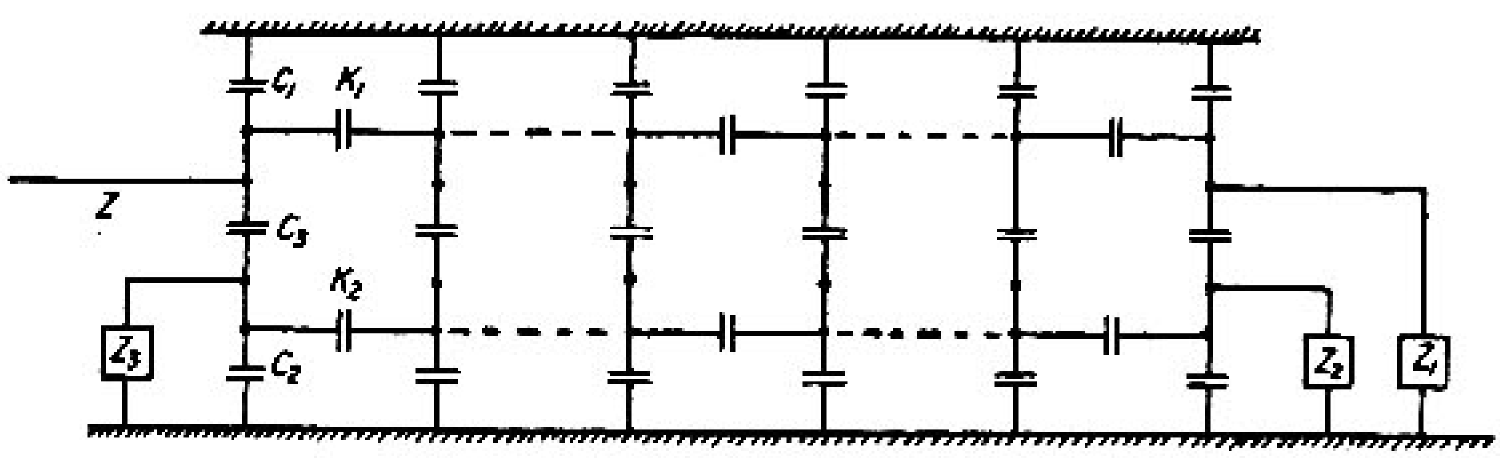


FIG. 99.—Circuit Determining the Initial Distributions

in sign, and is changing continually. It is therefore evident that any equations introduced to calculate the flux linkages must be simple enough so that they can be handled in complicated differential equations; but on the other hand they must be sufficiently complete to describe adequately the essential characteristics of the transient. The assumption that the flux linkages may be accounted for by a uniformly distributed self-inductance, as in transmission-line theory, proves entirely insufficient. The mutual inductance between parts of the same winding plays an important and essential part in the phenomenon. By most fortunate circumstances the assumption of a linearly graded mutual inductance between parts of the winding not only proves simple to handle mathematically, but also yields the essential characteristics to the transients.* Referring to Fig. 100, assume that the effective length of leakage path of any line of flux is $2h$, and that the coefficient of coupling between points in the primary and secondary windings equal distance from the end is σ. The assumption concerning the length of leakage path is equivalent to taking the coil stack in a long narrow slot open at both ends and extending a distance a either side of the coil stack, this extension

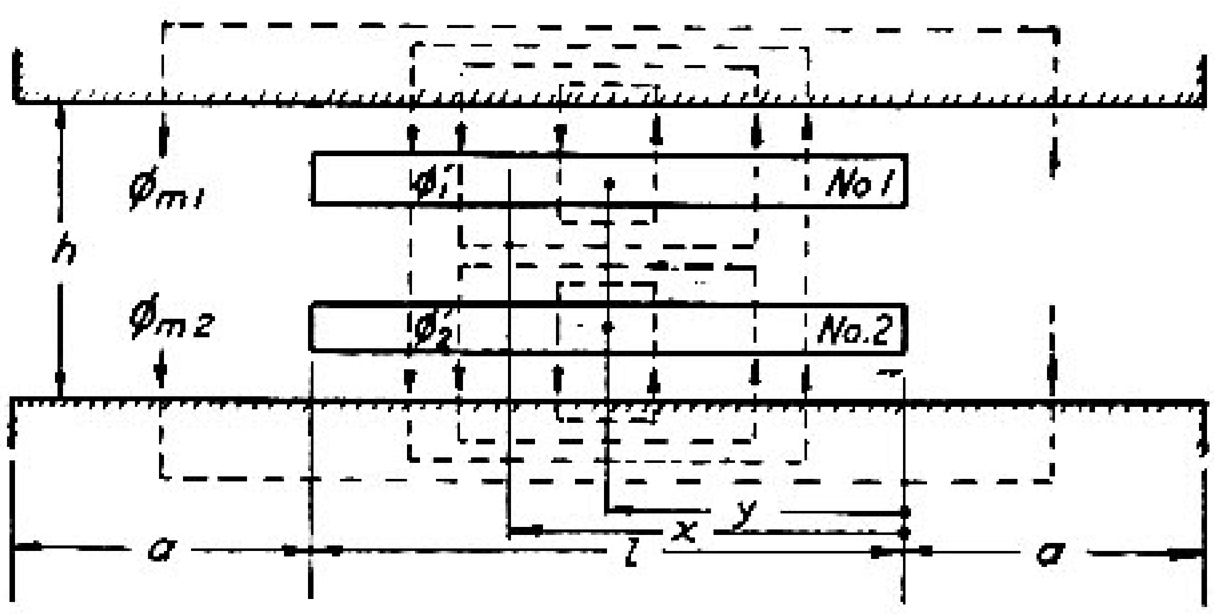


FIG. 100.—Flux Linkages in the Windings

being of sufficient length to account for the total mutual flux surrounding the entire windings.

Let $n_1 i_1$ = ampere-turns per unit length of winding 1.
$n_2 i_2$ = ampere-turns per unit length of winding 2.
$(m\,l\,t)$ = mean length of turn.

Then the mmf.'s at point y are

$$\left.\begin{array}{ll} 0.4\,\pi\,(n_1 i_1 + \sigma n_2 i_2)\,dy & \text{for the primary interlinkages.} \\ 0.4\,\pi\,(\sigma n_1 i_1 + n_2 i_2)\,dy & \text{for the secondary interlinkages.} \end{array}\right\}$$

The reluctance encountered by the flux due to these mmf.'s is

$$R = \frac{1}{(m\,l\,t)}\left[\frac{h}{a + l - y} + \frac{h}{a + y}\right] = \frac{h\,(2\,a + l)}{(a + l - y)\,(a + y)\,(m\,l\,t)}$$

* The method of taking the mutual and partial interlinkages into account, as given here, is substantially the same as that given by Blume and Boyajian, Reference 3 of the Bibliography.

and the fraction of this flux which links an element of winding at point x to the left of y is

$$\frac{a + l - x}{a + l - y}$$

The total flux linking x due to all the mmf. to the *right* of x therefore is (for the primary and secondary respectively)

$$\left.\begin{aligned} \phi_1' &= \frac{0.4\,\pi\,(m\,l\,t)}{(2\,a + l)\,h}\int_0^x (n_1\,i_1 + \sigma\,n_2\,i_2)\,(a + l - x)\,(a + y)\,dy \\ \phi_2' &= \frac{0.4\,\pi\,(m\,l\,t)}{(2\,a + l)\,h}\int_0^x (\sigma\,n_1\,i_1 + n_2\,i_2)\,(a + l - x)\,(a + y)\,dy \end{aligned}\right\} \quad (1a)$$

Likewise, the flux linking x due to all the mmf. to the *left* of x is

$$\left.\begin{aligned} \phi_1'' &= \frac{0.4\,\pi\,(m\,l\,t)}{(2\,a + l)\,h}\int_x^l (n_1\,i_1 + \sigma\,n_2\,i_2)\,(a + x)\,(a + l - y)\cdot dy \\ \phi_2'' &= \frac{0.4\,\pi\,(m\,l\,t)}{(2\,a + l)\,h}\int_x^l (\sigma\,n_1\,i_1 + n_2\,i_2)\,(a + x)\,(a + l - y)\,dy \end{aligned}\right\} \quad (1b)$$

Hence the total flux linkages with point x are

$$\left.\begin{aligned} \phi_1 &= \phi_1' + \phi_1'' \\ \phi_2 &= \phi_2' + \phi_2'' \end{aligned}\right\} \quad (2)$$

For example, if $n_1\,i_1$ and $n_2\,i_2$ are uniformly distributed, these terms may be taken out from under the integral signs, and then after integrating and simplifying

$$\left.\begin{aligned} \phi_1 &= \frac{0.4\,\pi\,(m\,l\,t)}{2\,h}\,(n_1\,i_1 + \sigma\,n_2\,i_2)\,[a\,l + x\,(l - x)] \\ \phi_2 &= \frac{0.4\,\pi\,(m\,l\,t)}{2\,h}\,(\sigma\,n_1\,i_1 + n_2\,i_2)\,[a\,l + x\,(l - x)] \end{aligned}\right\} \quad (3)$$

the term involving $a\,l$ being the *mutual* flux linkages and the term involving $x\,(l - x)$ being the *partial interlinkages*.

In addition to the transformer capacitance and flux constants discussed above, the circuit also involves series resistance and leakage conductance between coils and to ground. These loss factors are ignored in the present analysis since they greatly complicate the problem, and their influence is too small to change the character of the oscillations appreciably. In the subsequent chapter dealing with a single winding they are included, and found to be responsible for the introduction of exponential decrement factors, a slightly distorted final distribution, slightly changed frequencies of oscillation, and small reductions in the amplitudes of oscillations. The chief influence of the losses lies in the limitation which they impose on the magnitude of cumulative oscillations due to resonance, as is discussed in Chapter XIV.

GENERAL DIFFERENTIAL EQUATION

Referring to Fig. 101, the fundamental circuit equations are:

$$i_{k1} = K_1 \frac{\partial^2 e_1}{\partial x \, \partial t} \tag{4}$$

$$i_{k2} = K_2 \frac{\partial^2 e_2}{\partial x \, \partial t} \tag{5}$$

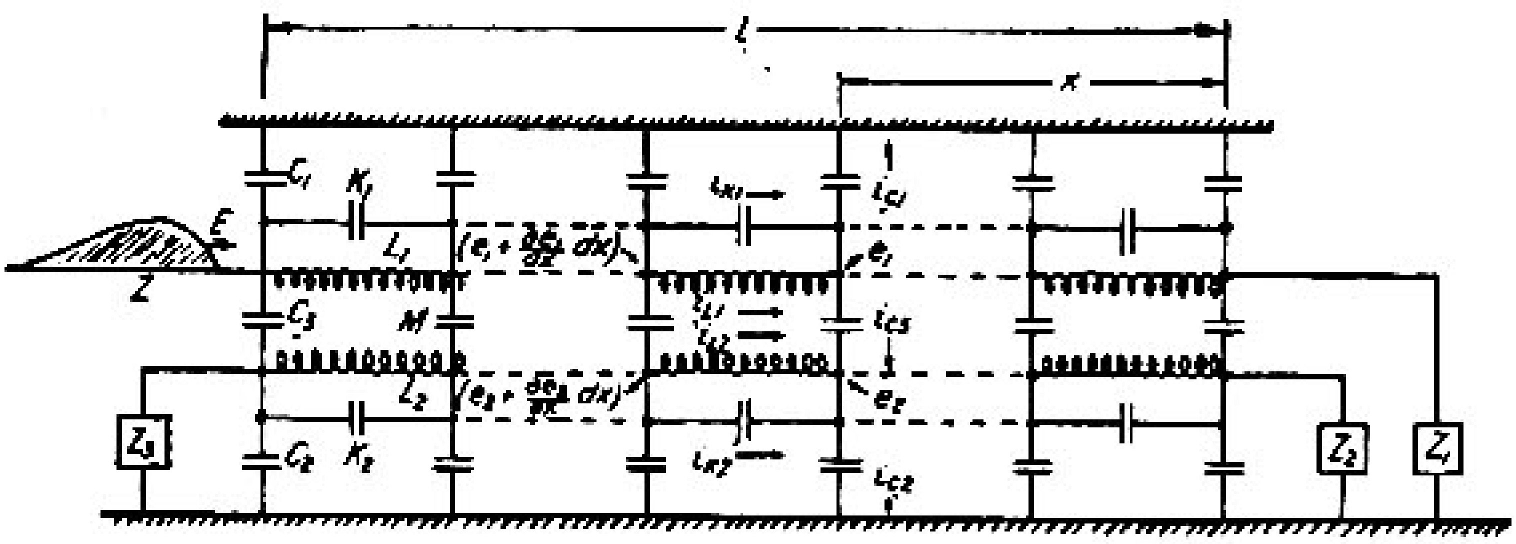


FIG. 101.—Complete Idealized Circuit of a Transformer to High-Frequency Transients

$$i_{c1} = C_1 \frac{\partial e_1}{\partial t} = \frac{\partial i_{k1}}{\partial x} + \frac{\partial i_{L1}}{\partial x} - i_{c3} \tag{6}$$

$$i_{c2} = C_2 \frac{\partial e_2}{\partial t} = \frac{\partial i_{k2}}{\partial x} + \frac{\partial i_{L2}}{\partial x} + i_{c3} \tag{7}$$

$$i_{c3} = C_3 \frac{\partial}{\partial t} (e_1 - e_2) \tag{8}$$

The fluxes of (2) induce voltage gradients

$$\frac{\partial e_1}{\partial x} = \frac{n_1}{10^8} \frac{\partial \phi_1}{\partial t} \tag{9}$$

$$\frac{\partial e_2}{\partial x} = \frac{n_2}{10^8} \frac{\partial \phi_2}{\partial t} \tag{10}$$

and differentiating three times to clear (2) of its integrals

$$\frac{\partial^4 e_1}{\partial x^4} = -\frac{0.4\,\pi\,(m\,l\,t)\,n_1}{10^8\,h}\,\frac{\partial^2}{\partial t\,\partial x}\,(n_1\,i_{L1} + \sigma\,n_2\,i_{L2})$$

$$= -\frac{\partial^2}{\partial t\,\partial x}\,(L_1\,i_{L1} + M\,i_{L2})\;l^3 \qquad (11)$$

$$\frac{\partial^4 e_2}{\partial x^4} = -\frac{\partial^2}{\partial t\,\partial x}\,(M\,i_{L1} + L_2\,i_{L2})\;l^3 \qquad (12)$$

in which

$$\left.\begin{aligned} L_1 &= \frac{0.4\,\pi\,n_1^2\,(m\,l\,t)\,l^3}{10^8\,h} \\ L_2 &= \frac{0.4\,\pi\,n_2^2\,(m\,l\,t)\,l^3}{10^8\,h} \\ M &= \frac{0.4\,\pi\,n_1\,n_2\,(m\,l\,t)\,\sigma\,l^3}{10^8\,h} \end{aligned}\right\} \qquad (13)$$

Hereafter the length of the winding will be taken as $l = 1$, and the circuit constants then pertain to the total length of the winding.

By (4), (5), (6), (7), and (8)

$$\frac{\partial^2}{\partial t\,\partial x}\,i_{L1} = (C_1 + C_3)\,\frac{\partial^2 e_1}{\partial t^2} - C_3\,\frac{\partial^2 e_2}{\partial t^2} - K_1\,\frac{\partial^4 e_1}{\partial x^2\,\partial t^2} \qquad (14)$$

$$\frac{\partial^2}{\partial t\,\partial x}\,i_{L2} = (C_2 + C_3)\,\frac{\partial^2 e_2}{\partial t^2} - C_3\,\frac{\partial^2 e_1}{\partial t^2} - K_2\,\frac{\partial^4 e_2}{\partial x^2\,\partial t^2} \qquad (15)$$

Substituting (14) and (15) into (11) and (12), respectively

$$\frac{\partial^4 e_1}{\partial x^4} - L_1\,K_1\,\frac{\partial^4 e_1}{\partial x^2\,\partial t^2} - M\,K_2\,\frac{\partial^4 e_2}{\partial x^2\,\partial t^2}$$

$$+(L_1\,C_1 + L_1\,C_3 - M\,C_3)\,\frac{\partial^2 e_1}{\partial t^2}$$

$$+(M\,C_2 + M\,C_3 - L_1\,C_3)\,\frac{\partial^2 e_2}{\partial t^2} = 0 \qquad (16)$$

$$\frac{\partial^4 e_2}{\partial x^4} - L_2\,K_2\,\frac{\partial^4 e_2}{\partial x^2\,\partial t^2} - M\,K_1\,\frac{\partial^4 e_1}{\partial x^2\,\partial t^2}$$

$$+(L_2\,C_2 + L_2\,C_3 - M\,C_3)\,\frac{\partial^2 e_2}{\partial t^2}$$

$$+(M\,C_1 + M\,C_3 - L_2\,C_3)\,\frac{\partial^2 e_1}{\partial t^2} = 0 \qquad (17)$$

From these two simultaneous differential equations, upon elimination there is, for either $e = e_1$ or $e = e_2$, the general differential equation

$$
\begin{aligned}
\frac{\partial^8 e}{\partial x^8} &- (L_1 K_1 + L_2 K_2) \frac{\partial^8 e}{\partial x^6 \partial t^2} + K_1 K_2 (L_1 L_2 - M^2) \frac{\partial^8 e}{\partial x^4 \partial t^4} \\
&+ [L_1 C_1 + L_2 C_2 + (L_1 + L_2 - 2 M) C_3] \frac{\partial^6 e}{\partial x^4 \partial t^2} \\
&- (L_1 L_2 - M^2) (K_1 C_2 + K_1 C_3 + K_2 C_1 + K_2 C_3) \frac{\partial^6 e}{\partial x^2 \partial t^4} \\
&+ (L_1 L_2 - M^2) (C_1 C_2 + C_2 C_3 + C_1 C_3) \frac{\partial^4 e}{\partial t^4} = 0 \qquad (18)
\end{aligned}
$$

This is the general differential equation, whose solution, subject to the boundary conditions imposed by the terminal impedances and the restriction imposed by auxiliary equations (16) and (17), yields the explicit equations of the transformer transients.

THE INITIAL DISTRIBUTION

The differential equation for the initial distribution can be obtained either by putting $\partial / \partial t = \infty$ in (18)—in accordance with the method of operational calculus—or by writing the differential equations directly from the capacitance network of Fig. 99, since at the first instant of impact of the traveling wave the current taken by the capacitance is infinite at that instant ($\partial E / \partial t = \infty$ for an abrupt rectangular wave front), whereas the current in the inductive paths is zero. Consequently the capacitance network determines the initial distribution.

Putting $\partial / \partial t = \infty$ in (18), there is

$$
\begin{aligned}
\frac{\partial^4 e}{\partial x^4} &- \left[\frac{K_1 (C_2 + C_3) + K_2 (C_1 + C_3)}{K_1 K_2}\right] \frac{\partial^2 e}{\partial x^2} \\
&+ \left[\frac{C_1 C_2 + C_2 C_3 + C_3 C_1}{K_1 K_2}\right] e = 0 \qquad (19)
\end{aligned}
$$

Putting $i_{L1} = i_{L2} = 0$ in (14) and (15), and canceling the operator $(\partial^2 / \partial t^2)$, there is

$$(C_1 + C_3)\, e_1 - K_1 \frac{\partial^2 e_1}{\partial x^2} = C_3\, e_2 \qquad (20)$$

$$(C_2 + C_3)\, e_2 - K_2 \frac{\partial^2 e_2}{\partial x^2} = C_3\, e_1 \qquad (21)$$

It is evident that (19) also follows from (20) and (21) upon eliminating either e_1 or e_2. The solution to (19), using (20), is

$$e_1 = P\epsilon^{\alpha x} + Q\epsilon^{-\alpha x} + R\epsilon^{\beta x} + S\epsilon^{-\beta x} \tag{22}$$

$$e_2 = P_2\epsilon^{\alpha x} + Q_2\epsilon^{-\alpha x} + R_2\epsilon^{\beta x} + S_2\epsilon^{-\beta x} \tag{23}$$

where

$$\left.\begin{aligned}
\alpha^2 &= \frac{\left\{\begin{matrix} K_1\,(C_2 + C_3) + K_2\,(C_1 + C_3) \\ +\sqrt{[K_1\,(C_2 + C_3) - K_2\,(C_1 + C_3)]^2 + 4\,K_1K_2C_3^2} \end{matrix}\right\}}{2\,K_1\,K_2} \\
\beta^2 &= \frac{\left\{\begin{matrix} K_1\,(C_2 + C_3) + K_2\,(C_1 + C_3) \\ -\sqrt{[K_1\,(C_2 + C_3) - K_2\,(C_1 + C_3)]^2 + 4\,K_1K_2C_3^2} \end{matrix}\right\}}{2\,K_1\,K_2}
\end{aligned}\right\} \tag{24}$$

The integration constants of (22) and (23) are related by both (20) and (21) as

$$\left.\begin{aligned}
P_2 &= P\,(C_1 + C_3 - K_1\,\alpha^2)\,/C_3 = m\,P \\
Q_2 &= Q\,(C_1 + C_3 - K_1\,\alpha^2)\,/C_3 = m\,Q \\
R_2 &= R\,(C_1 + C_3 - K_1\,\beta^2)\,/C_3 = n\,R \\
S_2 &= S\,(C_1 + C_3 - K_1\,\beta^2)\,/C_3 = n\,S
\end{aligned}\right\} \tag{25}$$

Thus the solutions for the initial distribution of both windings are of the same functional form, and the integration constants are related, so that there are only four integration constants to be determined from the terminal conditions. Referring to Fig. 99, there is

$$\left.\begin{aligned}
e_1 &= E, \quad \text{at } x = 1 \\
e_1 &= Z_1(p)\,i_{k1} = p\,Z_1(p)\,K_1\frac{\partial e_1}{\partial x}, \quad \text{at } x = 0 \text{ and } p = \infty \\
e_2 &= Z_2(p)\,i_{k2} = p\,Z_2(p)\,K_2\frac{\partial e_2}{\partial x}, \quad \text{at } x = 0 \text{ and } p = \infty \\
-\,e_2 &= Z_3(p)\,i_{k2} = p\,Z_3(p)\,K_2\frac{\partial e_2}{\partial x}, \quad \text{at } x = 1 \text{ and } p = \infty
\end{aligned}\right\} \tag{26}$$

The employment of these four conditions will yield four simultaneous equations in P, Q, R, S, which will suffice for their unique determination.

Case I. Grounded Neutral and Secondary Grounded at Both Terminals.—Here $Z_1 = Z_2 = Z_3 = 0$, and (26) yields

$$\left.\begin{aligned}
P\epsilon^{\alpha} + Q\epsilon^{-\alpha} + R\epsilon^{\beta} + S\epsilon^{-\beta} &= E \\
P + Q + R + S &= 0 \\
mP\epsilon^{\alpha} + mQ\epsilon^{-\alpha} + nR\epsilon^{\beta} + nS\epsilon^{-\beta} &= 0 \\
mP + mQ + nR + nS &= 0
\end{aligned}\right\} \tag{27}$$

Solving these four simultaneous equations,

$$\left.\begin{aligned}
P &= nE/(n-m)\,2\sinh\alpha \\
Q &= -nE/(n-m)\,2\sinh\alpha \\
R &= -mE/(n-m)\,2\sinh\beta \\
S &= mE/(n-m)\,2\sinh\beta
\end{aligned}\right\} \tag{28}$$

Hence by (22) and (23) the initial distributions (also expressed as half-range *sine* series) are

$$\begin{aligned}
\underset{t=0}{e_1} &= \frac{E}{n-m}\left[n\frac{\sinh\alpha x}{\sinh\alpha} - m\frac{\sinh\beta x}{\sinh\beta}\right] \\
&= \sum_{s=1}^{\infty}\frac{2E}{(m-n)}\left[\frac{ns\pi}{\alpha^2+s^2\pi^2} - \frac{ms\pi}{\beta^2+s^2\pi^2}\right]\cos s\pi\,\sin s\pi x
\end{aligned} \tag{29}$$

$$\begin{aligned}
\underset{t=0}{e_2} &= \frac{nmE}{n-m}\left[\frac{\sinh\alpha x}{\sinh\alpha} - \frac{\sinh\beta x}{\sinh\beta}\right] \\
&= \sum_{s=1}^{\infty}\frac{2\,nmE}{(m-n)}\left[\frac{s\pi}{\alpha^2+s^2\pi^2} - \frac{s\pi}{\beta^2+s^2\pi^2}\right]\cos s\pi\,\sin s\pi x
\end{aligned} \tag{30}$$

Case II. Grounded Neutral and Open-Circuited Secondary.—Here $Z_1 = 0$, $Z_2 = Z_3 = \infty$, and (26) yields

$$\left.\begin{aligned}
P\epsilon^{\alpha} + Q\epsilon^{-\alpha} + R\epsilon^{\beta} + S\epsilon^{-\beta} &= E \\
P + Q + R + S &= 0 \\
m\alpha P - m\alpha Q + n\beta R - n\beta S &= 0 \\
m\alpha P\epsilon^{\alpha} - m\alpha Q\epsilon^{-\alpha} + n\beta R\epsilon^{\beta} - n\beta S\epsilon^{-\beta} &= 0
\end{aligned}\right\} \tag{31}$$

Solving for the integration constants and substituting in (22) and (23)

$$\underset{t=0}{e_1} = E\frac{\begin{Bmatrix} mn\alpha\beta\ (\cosh\alpha - \cosh\beta)\ (\cosh\alpha x - \cosh\beta x) \\ + (m\alpha\sinh\alpha - n\beta\sinh\beta)\ (m\alpha\sinh\beta x - n\beta\sinh\alpha x) \end{Bmatrix}}{\begin{Bmatrix} 2\ mn\alpha\beta\ (1 - \cosh\alpha\cosh\beta) \\ + (m^2\alpha^2 + n^2\beta^2)\sinh\alpha\cdot\sinh\beta \end{Bmatrix}} \tag{32}$$

$$\underset{t=0}{e_2} = E\frac{\begin{Bmatrix} mn\alpha\beta\ (\cosh\alpha - \cosh\beta)\ (m\cosh\alpha x - n\cosh\beta x) \\ + mn\ (m\alpha\sinh\alpha - n\beta\sinh\beta)\ (\alpha\sinh\beta x - \beta\sinh\alpha x) \end{Bmatrix}}{\begin{Bmatrix} 2\ mn\alpha\beta\ (1 - \cosh\alpha\cdot\cosh\beta) \\ + (m^2\alpha^2 + n^2\beta^2)\sinh\alpha\cdot\sinh\beta \end{Bmatrix}} \tag{33}$$

Case III.—Isolated Neutral and Open-Circuited Secondary.— Here $Z_1 = Z_2 = Z_3 = \infty$, and (26) yields

$$\left.\begin{array}{llllll} P\,\epsilon^{\alpha} + & Q\,\epsilon^{-\alpha} + & R\,\epsilon^{\beta} + & S\,\epsilon^{-\beta} & = E \\ \alpha P & -\ \alpha Q & +\ \beta R & -\ \beta S & = 0 \\ m\,\alpha\,P & -\ m\,\alpha\,Q & +\ n\,\beta\,R & -\ n\,\beta\,S & = 0 \\ m\,\alpha\,P\,\epsilon^{\alpha} & -\ m\,\alpha\,Q\,\epsilon^{-\alpha} & +\ n\,\beta\,R\,\epsilon^{\beta} & -\ n\,\beta\,S\,\epsilon^{-\beta} & = 0 \end{array}\right\} \tag{34}$$

Solving for the integration constants and substituting in (22) and (23)

$$\underset{t=0}{e_1} = E\left\{\frac{m\,\alpha\sinh\alpha\cdot\cosh\beta\,x - n\,\beta\sinh\beta\cosh\alpha\,x}{m\,\alpha\sinh\alpha\cdot\cosh\beta - n\,\beta\sinh\beta\cosh\alpha}\right\} \tag{35}$$

$$\underset{t=0}{e_2} = nm\,E\left\{\frac{\alpha\sinh\alpha\cdot\cosh\beta\,x - \beta\sinh\beta\cosh\alpha\,x}{m\,\alpha\sinh\alpha\cdot\cosh\beta - n\,\beta\sinh\beta\cosh\alpha}\right\} \tag{36}$$

or, since $(\alpha\cdot\sinh\alpha)$ is large compared with $(\beta\cdot\sinh\beta)$ in practical cases, these expressions simplify to

$$\underset{t=0}{e_1} \cong E\frac{1}{(m\,\alpha - n\,\beta)}\left(\frac{m\,\alpha\cosh\beta\,x}{\cosh\beta} - \frac{n\,\beta\cosh\alpha\,x}{\cosh\alpha}\right) \tag{35a}$$

$$\underset{t=0}{e_2} \cong E\frac{nm}{(m\,\alpha - n\,\beta)}\left(\frac{\alpha\cosh\beta\,x}{\cosh\beta} - \frac{\beta\cosh\alpha\,x}{\cosh\alpha}\right) \tag{36a}$$

Case IV. Grounded Neutrals and Secondary Connected to Line.— Here $Z_1 = Z_2 = 0$, $Z_3 = z$, and (26) yields

$$\left.\begin{array}{lllll} P\epsilon^{\alpha} + & Q\epsilon^{-\alpha} + & R\epsilon^{\beta} + & S\epsilon^{-\beta} & = E \\ P\ + & Q\ + & R\ + & S & = 0 \\ mP\ + & mQ\ + & nR\ + & nS & = 0 \\ (1+a)\ mP\epsilon^{\alpha} + & (1-a)\ mQ\epsilon^{-\alpha} + & (1+b)\ nR\epsilon^{\beta} + & (1-b)\ nS\epsilon^{-\beta} & = 0 \end{array}\right\} \tag{37}$$

where $(a = z K_2 \alpha p)$ and $(b = z K_2 \beta p)$. Solving for the integration constants and substituting in (23)

$$e_2 = mn\left[\frac{(\sinh\beta + b\cosh\beta)\sinh\alpha x - (\sinh\alpha + a\cosh\alpha)\sinh\beta x}{n(\sinh\beta + b\cosh\beta)\sinh\alpha - m(\sinh\alpha + a\cosh\alpha)\sinh\beta}\right]E \quad (38)$$

At the terminal $x = 1$ this reduces to

$$e_2 = mn\left[\frac{\beta\sinh\alpha\cosh\beta - \alpha\sinh\beta\cosh\alpha}{n\beta\sinh\alpha\cdot\cosh\beta - m\alpha\sinh\beta\cosh\alpha}\right]\frac{p}{p+\gamma}E$$

$$= \frac{E\,mn\,(\beta\sinh\alpha\cosh\beta - \alpha\sinh\beta\cosh\alpha)}{(n\beta\sinh\alpha\cosh\beta - m\alpha\sinh\beta\cosh\alpha)}\,\epsilon^{-\gamma t} \quad (39)$$

where

$$\gamma = \frac{(n - m)\sinh\alpha\sinh\beta}{z K_2\,(n\beta\sinh\alpha\cosh\beta - m\alpha\sinh\beta\cosh\alpha)} \quad (40)$$

or approximately

$$e_2 \cong \frac{E\,mn.(\beta - \alpha)}{(n\beta - m\alpha)}\,\epsilon^{-\gamma t} \quad (39a)$$

$$\gamma \cong \frac{(n - m)}{z K_2\,(n\beta - m\alpha)} \quad (40a)$$

These equations hold rigorously only at $t = 0$, but under actual conditions this electrostatic transient is usually over within a fraction of a microsecond, and thus long before the electromagnetic transient due to the flow of current through the inductive paths has gained any headway. For the transformer constants given in the numerical example at the end of this Chapter, (39*a*) gives

$$e_2 = 0.193\,\epsilon^{-11.8t}$$

so that the time constant is less than a tenth of a microsecond. The crest value of this electrostatic transient may be comparable with that of the subsequent electromagnetic transient, depending, of course, on the relative capacitances, turn-ratio, and terminal connections of the transformer.

THE FINAL DISTRIBUTIONS

The realization of a final steady-state distribution at $t = \infty$ for an infinite rectangular applied wave is contingent upon the presence of losses, either in the transformer itself or in the terminal impedances, and the characteristics of such a final state will depend upon the nature of the losses. But in the no-loss circuit with zero or infinite terminal impedances, the axes of oscillation are determined by the electrostatic and electromagnetic fields necessary to establish the terminal voltages of the windings. Possible solutions to the general differential equation (18) which are independent of t and which satisfy the terminal conditions are:

$$e_1 = E_1' + x\,E_1 = e_n + (E - e_n)\,x \tag{41}$$

$$e_2 = E_2' + x\,E_2 \tag{42}$$

where e_n is the final neutral voltage ($e_n = 0$ for $Z_1 = 0$, $e_n = E$ for $Z_1 = \infty$).

It remains to identify these terms. From (4) to (8)

$$i_{L1} = I_1\,(t) + \int \left[(C_1 + C_3)\,\frac{\partial\, e_1}{\partial\, t} - C_3\,\frac{\partial\, e_2}{\partial\, t} - K_1\,\frac{\partial^3\, e_1}{\partial\, x^2\,\partial\, t} \right] dx \tag{43}$$

$$i_{L2} = I_2\,(t) + \int \left[(C_2 + C_3)\,\frac{\partial\, e_2}{\partial\, t} - C_3\,\frac{\partial\, e_1}{\partial\, t} - K_2\,\frac{\partial^3\, e_2}{\partial\, x^2\,\partial\, t} \right] dx \tag{44}$$

where $I_1\,(t)$ and $I_2\,(t)$ are integration constants with respect to x, and are therefore possible functions of t but not of x. Therefore I_1 and I_2 are common to all parts of their respective windings, and thus, in conjunction with the final electrostatic field, establish the final distributions or axes of oscillations. The indefinite integrals yield the space and time harmonics of the oscillation. Substituting I_1 and I_2 in (3) there follow by (9) and (10) the potential distributions caused by these currents.

$$\frac{n_1}{10^8}\int_0^x \frac{\partial\,\phi_1}{\partial\, t}\,dx$$

$$= x\,\frac{n_1}{10^8}\,\frac{\partial}{\partial\, t}\left[\Phi_{m1} + 0.2\,\pi\,(m\,l\,t)\,\frac{l}{h}\,(n_1\,I_1 + \sigma\,n_2\,I_2)\left(\frac{xl}{2} - \frac{x^2}{3}\right)\right] \tag{45}$$

$$n_2\int_0^x \frac{\partial\,\phi_2}{\partial\, t}\,dx$$

$$= x\,\frac{n_2}{10^8}\,\frac{\partial}{\partial\, t}\left[\Phi_{m2} + 0.2\,\pi\,(m\,l\,t)\,\frac{l}{h}\,(n_1\,I_1\,\sigma + n_2\,I_2)\left(\frac{xl}{2} - \frac{x^2}{8}\right)\right] \tag{46}$$

Now the mutual fluxes Φ_{m1} and Φ_{m2} which entirely surround the primary and secondary windings, respectively, are linear functions of

I_1 and I_2 and are large compared with the partial interlinkage terms, so that to a good approximation (45) and (46) become

$$x\,E_1 \cong x\frac{\partial}{\partial t}\,(L_1'\,I_1 + M'\,I_2) \tag{47}$$

$$x\,E_2 \cong x\frac{\partial}{\partial t}\,(M'\,I_1 + L_2'\,I_2) \tag{48}$$

Thus these two terms in (41) and (42) are due to a magnetic field. Writing $x = 1$ in (47) and (48), there results

$$E_1 = p\,(L_1'\,I_1 + M'\,I_2) = E - Z_1\,I_1 \tag{49}$$

$$E_2 = p\,(M'\,I_1 + L_2'\,I_2) = -\,(Z_2 + Z_3)\,I_2 \tag{50}$$

from which L_1', L_2', and M' are seen to be the overall self and mutual inductances of the windings, in the conventional sense. These two equations determine the currents I_1 and I_2 and therefrom the voltages E_1 and E_2 and the terminal transients $Z_1\,I_1$, $Z_2\,I_2$, and $-Z_3\,I_2$.

Case I. Grounded Neutral and Short-Circuited Secondary.—Here $Z_1 = Z_2 = Z_3 = 0$, $E_2 = 0$, and $E_1 = E$. Hence by (49) and (50):

$$\left.\begin{aligned} I_1 &= \frac{E\,L_2'\,t}{(L_1'\,L_2' - M'^2)} \\ I_2 &= \frac{E\,M'\,t}{(L_1'\,L_2' - M'^2)} \end{aligned}\right\} \tag{51}$$

Thus the currents increase linearly with time at a rate sufficient to consume the primary applied voltage E.

Case II. Grounded Neutral and Open-Circuited Secondary.—Here $Z_1 = 0$, $I_2 = 0$, $E_1 = E$. Hence by (49) and (50)

$$\left.\begin{aligned} I_1 &= \frac{E}{L_1'}\,t \\ E_2 &= \frac{M'}{L_1'}\,E \cong \frac{n_2}{n_1}\,E \end{aligned}\right\} \tag{52}$$

Case III. Isolated Primary Neutral.—Here

$$I_1 = 0, \quad I_2 = 0, \quad E_1 = 0, \quad \text{and} \quad E_2 = 0 \tag{53}$$

The more general case with impedances at the neutral and secondary terminals is reserved for Chapter XV under the heading "Terminal Transients." The currents I_1 and I_2 of (49) and (50) very soon greatly exceed the electrostatic and oscillatory components of the transient currents (for practical terminal impedances), and it is these currents, therefore, which not only fix the axes of oscillation but also dominate the terminal transients.

Electrostatic Component of the Final Distribution.—The total dielectric flux which enters the secondary must be balanced by that which leaves it. Let C_2' and C_3' be the capacitances to ground, under steady-state conditions, of the terminal impedances Z_2 and Z_3 respectively. Then the total flux entering the secondary from the primary is, using (41) and (42),

$$\psi = \int_0^1 C_3 \left[e_n + (E - e_n)\, x - E_2' - x\, E_2\right] dx \tag{54}$$

and this must be equal to the total dielectric flux which leaves the secondary by way of C_2, C_2' and C_3'. It is

$$\psi = C_2' E_2' + C_2 \int_0^1 (E_2' + x\, E_2)\, dx + C_3' (E_2 + E_2') \tag{55}$$

Equating (54) and (55), there results

$$E_2' = \frac{C_3 (E + e_n) - (C_2 + C_3 + 2\, C_3')\, E_2}{2\,(C_2 + C_3 + C_2' + C_3')} \tag{56}$$

which identifies the corresponding term in (42) as the potential of the secondary due to its position in the electrostatic field.

If the secondary neutral end is grounded (directly or through a resistance or inductance), $C_2' = \infty$, and (56) gives $E_2' = 0$.

If the secondary line end is grounded, $C_3' = \infty$, and (56) gives $E_2' = -E_2$.

If the primary neutral is isolated, $e_n = E$ and $E_2 = 0$, and (56) gives $E_2' = C_3\, E\ \ (C_2 + C_3 + C_2' + C_3')$.

In an actual transformer with isolated secondary terminals, the leakage conductance will bring the final average potential to zero.

SOLUTION OF THE DIFFERENTIAL EQUATION

The solution must satisfy the differential equation, the terminal conditions, and the initial and final distributions. As a tentative solution for directly grounded or open-circuited terminal conditions, assume

$$e_1 = e_n + (E - e_n)\, x + \sum A\ \epsilon^{(ax+bt)} \tag{57}$$

$$e_2 = E_2' + E_2\, x + \sum A_2\ \epsilon^{(ax+bt)} \tag{58}$$

in which the terms outside the summation will be recognized as the final distributions, (41) and (42), and A and A_2 are integration constants. Substituting (57) or (58) in (18), and simplifying, there results

$$(L_1 L_2 - M^2)\,[K_1K_2\,a^4 - (K_1C_2 + K_2C_1 + K_1C_3 + K_2C_3)\,a^2 + (C_1C_2 + C_2C_3 + C_3C_1)]\,b^4 + [-(L_1K_1 + L_2K_2)\,a^6 + (L_1C_1 + L_2C_2 + L_1C_3 + L_2C_3 - 2\,M\,C_3)\,a^4]\,b^2 + a^8 = 0 \quad (59)$$

This equation is a quadratic in b^2, of which the coefficients are all positive if a is a pure imaginary $\pm j\,\lambda$, because $L_1 L_2 \geq M^2$ and $(L_1 + L_2) \geq 2\,M$. Moreover, upon expanding the terms under the radical of the solution for b^2, it becomes evident that the radical is a positive real number, and accordingly b^2 has two real negative roots. Corresponding to these two negative values, there is

$$b = \begin{cases} \pm j\,\omega \\ \pm j\,\Omega \end{cases} \quad (60)$$

Thus if a is imaginary there are two corresponding imaginary values for b, and (57) and (58) become oscillatory. This is a necessary consequence of the fact that the circuit is a nondissipative network of inductances and capacitances, and therefore exponential decrement factors can not appear in its solution. The solutions may therefore be written in the form

$$\begin{aligned} e_1 = e_n + (E - e_n)\,x + \textstyle\sum[&(A \cos \omega t + A' \cos \Omega\,t) \sin \lambda\,x \\ &+ (B \sin \omega t + B' \sin \Omega\,t) \sin \lambda\,x \\ &+ (C \cos \omega t + C' \cos \Omega\,t) \cos \lambda\,x \\ &+ (D \sin \omega t + D' \sin \Omega\,t) \cos \lambda\,x] \end{aligned} \quad (61)$$

$$\begin{aligned} e_2 = E_2' + x\,E_2 \quad + \textstyle\sum[&(A_2 \cos \omega t + A_2' \cos \Omega\,t) \sin \lambda\,x \\ &+ (B_2 \sin \omega t + B_2' \sin \Omega\,t) \sin \lambda\,x \\ &+ (C_2 \cos \omega t + C_2' \cos \Omega\,t) \cos \lambda\,x \\ &+ (D_2 \sin \omega t + D_2' \sin \Omega\,t) \cos \lambda\,x] \end{aligned} \quad (62)$$

Thus there are *two* time harmonics to each space harmonic. In practical cases only one of each pair of time harmonics is important in the primary winding, but both are important in the secondary. There are nineteen constants to be determined in (61) and (62)—the sixteen integration constants, the wave length constant λ, and the angular velocities ω and Ω. Of these, ω and Ω are given by (59) and (60) as soon as λ is known. Furthermore, the integration constants of the secondary are related to those of the primary through the auxiliary equations (16) and (17). Substituting (61) and (62) in (16) and separately equating the coefficients of like trigonometric terms, there results

$$\frac{A_2}{A} = \frac{B_2}{B} = \frac{C_2}{C} = \frac{D_2}{D} = r_s$$

$$= \frac{\lambda^4 - \omega^2 (\lambda^2 L_1 K_1 + L_1 C_1 + L_1 C_3 - M C_3)}{\omega^2 (\lambda^2 M K_2 + M C_2 + M C_3 - L_1 C_3)} \tag{63}$$

$$\frac{A_2'}{A'} = \frac{B_2'}{B'} = \frac{C_2'}{C'} = \frac{D_2'}{D'} = r_s'$$

$$= \frac{\lambda^4 - \Omega^2 (\lambda^2 L_1 K_1 + L_1 C_1 + L_1 C_3 - M C_3)}{\Omega^2 (\lambda^2 M K_2 + M C_2 + M C_3 - L_1 C_3)} \tag{64}$$

If (61) and (62) are substituted in (17) instead of (16) there results

$$r_s = \frac{\omega^2 (\lambda^2 M K_1 + M C_1 + M C_3 - L_2 C_3)}{\lambda^4 - \omega^2 (\lambda^2 L_2 K_2 + L_2 C_2 + L_2 C_3 - M C_3)} \tag{65}$$

$$r_s' = \frac{\Omega^2 (\lambda^2 M K_1 + M C_1 + M C_3 - L_2 C_3)}{\lambda^4 - \Omega^2 (\lambda^2 L_2 K_2 + L_2 C_2 + L_2 C_3 - M C_3)} \tag{66}$$

That (65) is equal to (63), and (66) equal to (64), is evident through (59), and therefore it is immaterial which pair of expressions be used.

At $t = 0$, (61) and (62) must be equal to the initial distributions (22) and (23) respectively. Therefore, making these substitutions and expressing the *difference* between the initial distribution and the final distribution as a Fourier series, there is

$$\Sigma[(A + A') \sin \lambda x + (C + C') \cos \lambda x]$$
$$= (P \epsilon^{\alpha x} + Q \epsilon^{-\alpha x} + R \epsilon^{\beta x} + S \epsilon^{-\beta x}) - [e_n + (E - e_n) x]$$
$$= \sum_{s=1}^{\infty} \left(X_s \sin \frac{s\pi x}{c} + Y_s \cos \frac{s\pi x}{c} \right) \tag{67}$$

$$\Sigma[(r_s A + r_s' A') \sin \lambda x + (r_s C + r_s' C') \cos \lambda x]$$
$$= (m P \epsilon^{\alpha x} + m Q \epsilon^{-\alpha x} + n R \epsilon^{\beta x} + n S \epsilon^{-\beta x}) - (E_2' + x E_2)$$
$$= \sum_{s=1}^{\infty} \left(U_s \sin \frac{s\pi x}{c} + V_s \cos \frac{s\pi x}{c} \right) \tag{68}$$

where X_s, Y_s, U_s, and V_s are the coefficients in the Fourier series of the *difference* between the *initial* and *final* distributions; and c is the half wave length upon which the Fourier analysis is made. The choice of this wave length depends upon the terminal conditions, and it is possible to find a suitable value only for certain conditions, which means that a Fourier series is not always an appropriate type of expression applicable to any terminal conditions. Comparing the coefficients of like trigonometric terms in (67) and (68), it is seen that

$$\left.\begin{aligned} A + \quad A' &= X_s \\ C + \quad C' &= Y_s \\ r_s A + r_s' A' &= U_s \\ r_s C + r_s' C' &= V_s \\ \lambda &= s\pi/c \end{aligned}\right\} \tag{69}$$

and solving these simultaneous equations

$$\left.\begin{aligned} A &= (r_s' X_s - U_s) \; (r_s' - r_s) \\ A' &= (r_s X_s - U_s) \; (r_s - r_s') \\ C &= (r_s' Y_s - V_s) \; (r_s' - r_s) \\ C' &= (r_s Y_s - V_s) \; (r_s - r_s') \end{aligned}\right\} \tag{70}$$

The solution must satisfy the following terminal conditions:

$$i_{L1} = i_{L2} = 0 \text{ at } t = 0 \tag{71}$$

$$\left.\begin{aligned} (i_{L1} + i_{K1})\, Z_1 &= e_1 \text{ at } x = 0 \\ (i_{L2} + i_{K2})\, Z_2 &= e_2 \text{ at } x = 0 \\ -(i_{L2} + i_{K2})\, Z_3 &= e_2 \text{ at } x = 1 \end{aligned}\right\} \tag{72}$$

To impose the first of these conditions, substitute (61) and (62) into (43), perform the indicated operations, and equate to zero at $t = 0$. Then, since $I_1(t) = 0$ at $t = 0$, the coefficients of like trigonometric terms may be equated to zero, yielding

$$\frac{B'}{B} = \frac{D'}{D} = -\frac{\omega}{\Omega}\left[\frac{(C_1 + C_3 + K_1\lambda^2) - r\,C_3}{(C_1 + C_3 + K_1\lambda^2) - r'\,C_3}\right] \tag{73}$$

Making the same substitution in (44) there results

$$\frac{B'}{B} = \frac{D'}{D} = -\frac{\omega}{\Omega}\left[\frac{(C_2 + C_3 + K_2\lambda^2)\,r - C_3}{(C_2 + C_3 + K_2\lambda^2)\,r' - C_3}\right] \tag{74}$$

But the right-hand members of (73) and (74) are different, and therefore the only value for the integration constants which satisfies these relationships is

$$B = B' = D = D' = 0 \tag{75}$$

In other words, (73) and (74) are the equations of straight lines which intersect at the origin of coordinates B' and B (or D' and D), and therefore $(B', B) = (0, 0)$ is the common point which satisfies both graphs. The general solutions (61) and (62) now become (dropping subscripts)

$$\begin{aligned} e_1 = e_n + (E - e_n)\, x + \sum_{s=1}^{\infty} [(A \cos \omega t + A' \cos \Omega t) \sin \lambda x \\ + (C \cos \omega t + C' \cos \Omega t) \cos \lambda x] \end{aligned} \tag{76}$$

$$\begin{aligned} e_2 = E_2' + x\, E_2 \qquad + \sum_{s=1}^{\infty} [(r\, A \cos \omega t + r'\, A' \cos \Omega t) \sin \lambda x \\ + (r\, C \cos \omega t + r'\, C' \cos \Omega t) \cos \lambda x] \end{aligned} \tag{77}$$

where

$$\left.\begin{array}{ll} \lambda = s\pi c & \text{from (69)} \\ \omega, \Omega, & \text{from (59) and (60)} \\ A, A', C, C' & \text{from (70)} \\ r, r' & \text{from (65) and (66)} \\ E_2' & \text{from (56)} \\ E_2 & \text{from (50)} \\ c & \text{must satisfy (72) if (76) and (77)} \\ & \quad \text{are possible solutions} \end{array}\right\} \tag{78}$$

The last term under the integral of (43) is i_{K1}, and the last term under the integral of (44) is i_{K2}. Therefore, upon rearrangement

$$(i_{L1} + i_{K1}) = \int \left[(C_1 + C_3) \frac{\partial e_1}{\partial t} - C_3 \frac{\partial e_2}{\partial t} \right] dx + I_1(t) \tag{79}$$

$$(i_{L2} + i_{K2}) = \int \left[(C_2 + C_3) \frac{\partial e_2}{\partial t} - C_3 \frac{\partial e_1}{\partial t} \right] dx + I_2(t) \tag{80}$$

where $I_1(t)$ and $I_2(t)$ are integration constants with respect to x, and are therefore possible functions of time. They are the same terms which appeared in (43) and (44).

At a grounded terminal the voltage must be zero; at an open terminal the currents, as given by (79) or (80), must be zero. Upon substituting (61) and (62) in (79) or (80), it is seen that voltage harmonics distributed as sin λx (or cos λx) yield current harmonics which are distributed as cos λx (or sin λx). Therefore cases arise for which the same space harmonics will not satisfy both the primary and secondary terminal conditions. However, the two cases of Fig. 102 are satisfied by the same Fourier expansion for both primary and secondary, as indicated. In these circuits the primary line terminal is shown grounded, because its actual potential E is accounted for by the final distribution term in the general solution, and therefore the line terminal is quiescent with respect to harmonic oscillations of voltage. The expansions for the two circuits of Fig. 102 are

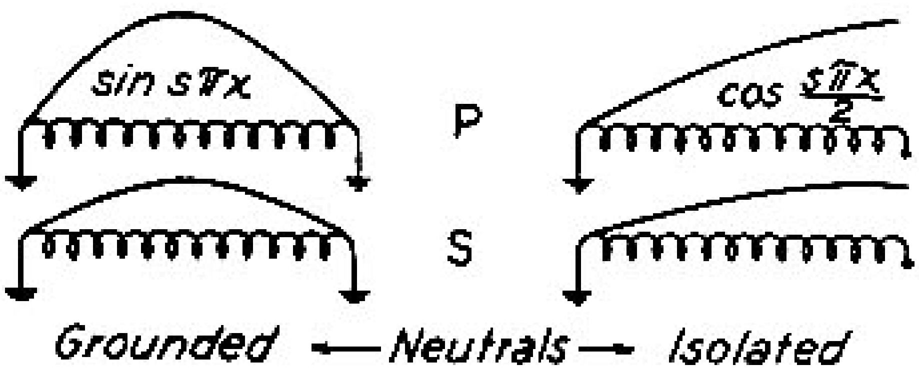


FIG. 102.—Fundamental of Voltage Distributions

Neutral	Grounded	Isolated
Z_1	0	∞
Z_2	0	∞
Z_3	0	0
c	1	2
Primary	X_s	Y_{2s-1}
Secondary	U_s	V_{2s-1}

The subscripts s and $2s - 1$ indicate that *all* harmonics (both even and odd) are present for grounded neutral, whereas only the *odd* harmonics are present for isolated neutral.

Numerical Example.—Consider a transformer with the following constants

$$C_1 = 1 \times 10^{-9} \qquad K_1 = 1 \times 10^{-11} \qquad L_1 = 10$$

$$C_2 = 2 \times 10^{-9} \qquad K_2 = 1 \times 10^{-11} \qquad L_2 = 2.5$$

$$C_3 = 1 \times 10^{-9} \qquad M = 2.5$$

with grounded neutral, and the secondary short-circuited and grounded. By Fig. 102 it is seen that the appropriate expansion is the half-range *sine* series, $c = 1$ and $\lambda = s\pi$.

By (24) and (25)

$$\alpha^2 = \frac{3 + 2 + \sqrt{1 + 4}}{0.02} = 361.7, \qquad \alpha = 19.0$$

$$\beta^2 = \frac{3 + 2 - \sqrt{1 + 4}}{0.02} = 138.2 \qquad \beta = 11.7$$

$$m = 2 - 3.617 \qquad = -1.617$$

$$n = 2 - 1.382 \qquad = +0.618$$

By (29) the initial distributions are

$$\underset{t=0}{e_1} = E\left[0.276\,\frac{\sinh 19.0\,x}{\sinh 19.0} + 0.724\,\frac{\sinh 11.7\,x}{\sinh 11.7}\right]$$

$$= \sum_{s=1}^{\infty}\left[\frac{-0.552\,s\pi}{361.7 + s^2\pi^2} + \frac{1.448\,s\pi}{138.2 + s^2\pi^2}\right]\cos s\pi\,\sin s\pi x$$

$$\underset{t=0}{e_2} = 0.447\,E\left[\frac{\sinh 19.0\,x}{\sinh 19.0} - \frac{\sinh 11.7\,x}{\sinh 11.7}\right]$$

$$= \sum_{s=1}^{\infty}\left[\frac{-0.894\,s\pi}{361.7 + s^2\pi^2} + \frac{0.894\,s\pi}{138.2 + s^2\pi^2}\right]\cos s\pi\,\cos s\pi x$$

By (50) and (56) the terms of the final distribution are

$$\underset{t=\infty}{e_1} = e_n + (E - e_n)\,x = 0 + x\,E = \sum_{1}^{\infty}\frac{-2\,E\cos s\pi}{s\pi}\sin s\pi x$$

$$\underset{t=\infty}{e_2} = E_2' + x\,E_2 \qquad = 0$$

By (59) and (60)

$$b^2 = \frac{\left\{\begin{array}{l} -(0.125\,s^6\pi^6 + 22.5\,s^4\pi^4) \pm 9\,s^4\pi^4 \\ \sqrt{(s^4\pi^4\,10^4 + 2.31\,s^2\pi^2\,100 + 1.615)}\end{array}\right\}}{37.5\left(\frac{s^4\pi^4}{10^4} + 5\,\frac{s^2\pi^2}{10^2} + 5\right)10^{-9}}$$

$$= \begin{cases} \omega^2 \text{ for the } + \text{ sign} \\ \Omega^2 \text{ for the } - \text{ sign}\end{cases}$$

and by (63) and (64)

$$r_s' = \frac{s^4\pi^4 - \Omega^2\,(s^2\pi^2 + 175)\,10^{-10}}{0.25\,\Omega^2\,(s^2\pi^2 - 100)\,10^{-10}}$$

$$r_s = \frac{s^4\pi^4 - \omega^2\,(s^2\pi^2 + 175)\,10^{-10}}{0.25\,\omega^2\,(s^2\pi^2 - 100)\,10^{-10}}$$

By (67) and (68)

$$X_s = \left[\frac{-0.552\, s\pi}{361.7 + s^2\pi^2} + \frac{1.448\, s\pi}{138.2 + s^2\pi^2} + \frac{2}{s\pi}\right] \cos s\pi$$

$$U_s = \left[\frac{-0.894\, s\pi}{361.7 + s^2\pi^2} + \frac{0.894\, s\pi}{138.2 + s^2\pi^2} - 0\right] \cos s\pi$$

$$V_s = 0$$

By (70)
$$A = \frac{r_s' X_s - U_s}{r_s' - r_s}, \qquad C = 0$$

$$A' = \frac{r_s X_s - U_s}{r_s - r_s'}, \qquad C' = 0$$

The solutions then are, by (76) and (77)

$$e_1 = x E + \sum_1^{\infty} (A_s \cos \omega_s t + A_s' \cos \Omega_s t) \sin s\pi x$$

$$e_2 = 0 + \sum_1^{\infty} (A_s r_s \cos \omega_s t + A_s' r_s' \cos \Omega_s t) \sin s\pi x$$

The following table, calculated by H. L. Rorden, gives the numerical results for the above case:

Short-Circuited and Grounded Secondary			
s	1	2	3
ω_s Ω_s	132,000 74,500	500,000 276,000	1,140,000 553,000
r_s r_s'	5.65 0.216	10.08 0.308	81.3 0.88
X_s U_s	− 0.592 0.011	0.252 − 0.018	− 0.138 0.018
A_s A_s'	0.0254 − 0.617	− 0.010 0.259	0.002 − 0.139
A_{s2} A_{s2}'	0.144 − 0.133	− 0.097 0.079	0.141 − 0.123
A_s * $2\pi f$ *	− 0.601 74,700	0.257 278,000	− 0.138 563,000

* Based on single-winding theory given in Chapter XIII.

SUMMARY OF CHAPTER XII

The idealized circuit of the two-winding transformer is characterized by an eighth-order partial differential equation. Under certain terminal conditions, solutions are obtainable, the salient features of which are:

1. The initial distributions and the electrostatic components of the terminal transients, determined entirely by the capacitances of the windings and the terminal impedances.
2. The final distributions, or axes of oscillations, and the electromagnetic components of the terminal transients, determined by the inductances of the winding, the final steady-state electrostatic fields, and the terminal impedances.
3. The internal transient oscillations, which consist of an infinite series of space and time harmonics oscillating about the final distributions as axes of oscillations, and whose amplitudes depend upon a Fourier analysis of the difference between the initial and final distributions. To each space harmonic there correspond two time harmonics. Only one of these two sets of time harmonics is of importance in the primary oscillation, and the transient associated therewith is substantially the same as that yielded by the much simpler single-winding theory discussed in the next chapter. But both sets of time harmonics are of practically equal importance in the secondary, and since they are initially opposite in phase, the initial distribution of the secondary is no indication of the severity of the oscillations which may occur therein.
4. The practical utility of the complicated two-winding theory lies principally in the fact that it rigorously establishes the validity of the single-winding approximation, and the simplified equivalent circuits for terminal transients discussed in a subsequent chapter. In addition, of course, it provides the only means available for calculating the internal oscillations of the secondary.

CHAPTER XIII

TRANSIENT OSCILLATIONS IN THE PRIMARY WINDINGS *

Numerical calculations covering practical cases, based on the analysis given in the previous chapter, show that the essential characteristics of the oscillations in the primary winding are substantially the same as obtain when the secondary winding is ignored, provided that the Fourier expansion is on the same base in both cases. It is appropriate, therefore, to give the analysis for a single independent winding, because the equations then become greatly simplified and easy to visualize, and a number of characteristic curves can be prepared. It must be borne in mind, however, that it may be necessary to make arbitrary changes in the circuit constants, particularly of the inductance coefficient, to obtain accurate numerical agreement. Also, the minor frequency set disappears in the single-winding theory, and there is no explicit indication of the effect of the secondary in fixing the appropriate Fourier expansion.

The following analysis is along the same lines as that given for the two-winding theory, and is idealized to the same extent, but the effect of the losses and of the applied wave shape is taken into account, and the influence of each of the several circuit constants is discussed. As before, the derivations are restricted to either a grounded or isolated neutral.

THE GENERAL DIFFERENTIAL EQUATION

Referring to Fig. 103*A*, the circuit constants per unit length of winding are:

L = inductance coefficient, including the partial interlinkages.
$M(x, y)$ = mutual inductance between elements at x and y.
C = shunt capacitance to ground.
K = series capacitance along the winding.
G = shunt conductance to ground.

* "Transient Oscillations in Distributed Circuits with Special Reference to Transformer Windings," by L. V. Bewley, *A.I.E.E. Trans.*, Vol. 50, 1931.

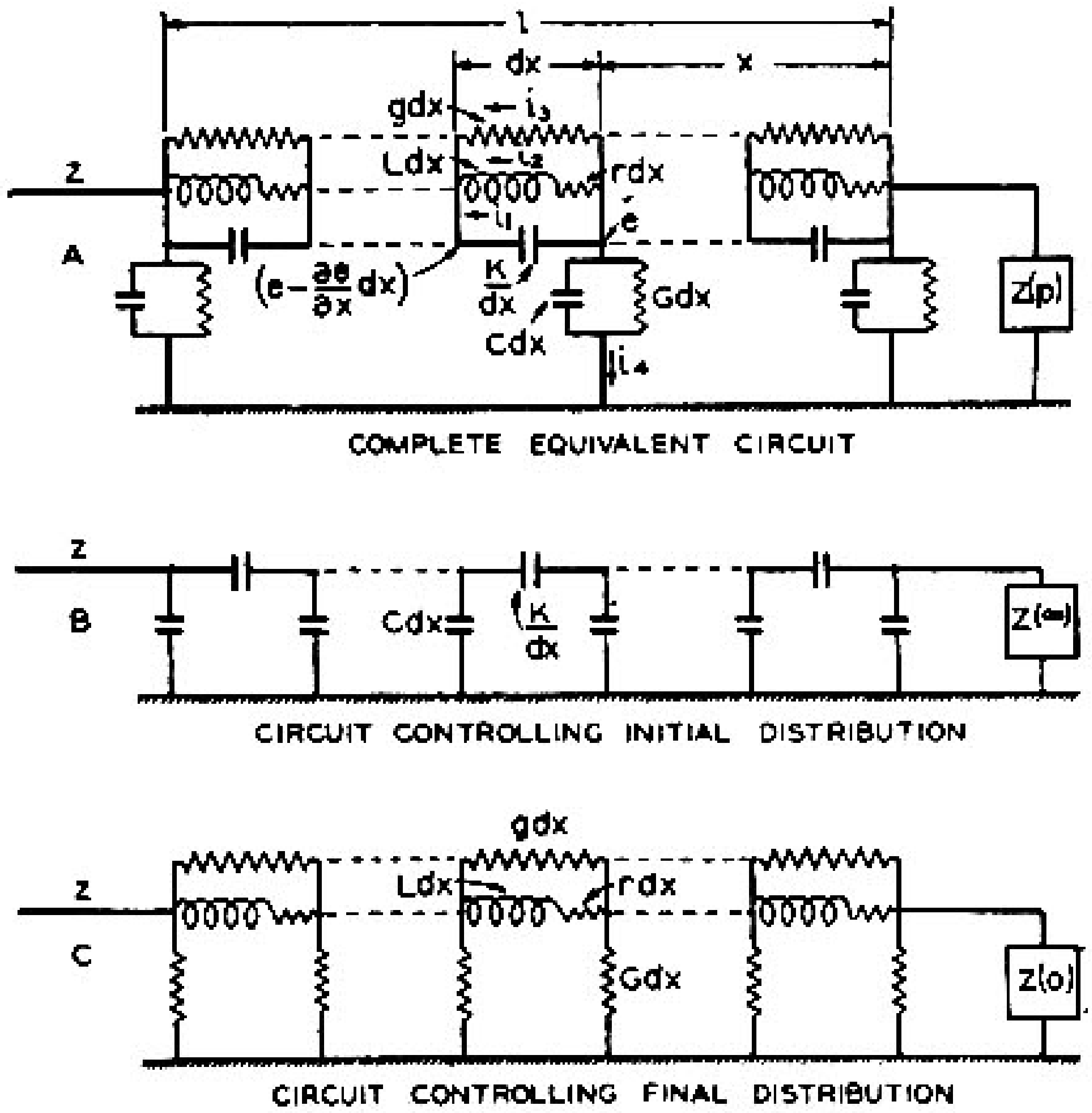


FIG. 103.—Ideal Complete Circuit of a Winding

g = shunt inductance along the winding.
r = series resistance.
n = turns.

The variables involved at any point of the winding are:

e = potential to ground.
i_1 = current in series capacitance K.
i_2 = current in the inductance L.
i_3 = current in the shunt conductance g.
i_4 = current to ground through G and C.
ϕ = total flux linkages at a point.
B = flux density.
t = time.
$p = \partial / \partial t$ = partial derivative with respect to time.
x, y = points along the winding, measured from the neutral end.
l = length of the winding.
$(m\,l\,t)$ = mean length of turn.
$2\,h$ = length of the leakage path.

The fundamental relationships are:

$$i_1 = K \frac{\partial^2 e}{\partial x \, \partial t} \tag{1}$$

$$i_3 = g\frac{\partial e}{\partial x} \tag{2}$$

$$i_4 = \left(G + C\frac{\partial}{\partial t}\right)e = \frac{\partial}{\partial x}(i_1 + i_2 + i_3) \tag{3}$$

$$\frac{\partial e}{\partial x} = r\, i_2 + \frac{n}{10^8}\frac{\partial \phi}{\partial t} \tag{4a}$$

$$= r\, i_2 + \frac{\partial}{\partial t}\int_0^l M\,(x, y)\; i_2\,(y)\cdot d\,y \tag{4b}$$

$$= r\, i_2 + \frac{\partial}{\partial t}\Big\{L'\, i_2\,(x)$$

$$+ \int_0^l M\,(x, y)\,[i_2\,(y) - i_2\,(x)]\,d\,y\Big\} \tag{4c}$$

where

$$L' = \int_0^l M\,(x, y)\,d\,y = \text{self inductance}$$

and as in (2) of Chapter XII

$$\phi = \phi_m + \phi_l = \phi_m + \frac{0.4\,\pi\,(m\,l\,t)\,n}{h}\int_0^x\int_x^l i_2\,dy\,dz \tag{5}$$

where

ϕ_m = flux mutual to the entire winding.

ϕ_l = flux due to partial interlinkages.

From (4*a*) and (5)

$$\frac{\partial^4 e}{\partial x^4} = r\frac{\partial^3 i_2}{\partial x^3} - \frac{0.4\,\pi\,n^2\,(m\,l\,t)}{h\,10^8}\,\frac{\partial^2 i_2}{\partial x\,\partial t}$$

$$= r\frac{\partial^3 i_2}{\partial x^3} - \frac{L}{l^3}\frac{\partial^2 i_2}{\partial x\,\partial t} \tag{6}$$

where

$$L = \frac{0.4\,\pi\,n^2\,l^3\,(m\,l\,t)}{h\,10^8} = \text{effective inductance}$$

By (1), (2), and (3) there is

$$\frac{L}{l^3}\frac{\partial^2 i_2}{\partial x\,\partial t} = \frac{L}{l^3}\left(G + C\frac{\partial}{\partial t}\right)\frac{\partial e}{\partial t} - \frac{L}{l^3}\frac{\partial^2 i_1}{\partial x\,\partial t} - \frac{L}{l^3}\frac{\partial^2 i_3}{\partial x\,\partial t}$$

$$= \frac{L}{l^3}G\frac{\partial e}{\partial t} + \frac{L}{l^3}C\frac{\partial^2 e}{\partial t^2} - \frac{L}{l^3}K\frac{\partial^4 e}{\partial x^2\,\partial t^2} - g\frac{L}{l^3}\frac{\partial^3 e}{\partial x^2\,\partial t} \tag{7}$$

and

$$r \frac{\partial^3 i_2}{\partial x^3} = r G \frac{\partial^2 e}{\partial x^2} + r C \frac{\partial^3 e}{\partial x^2 \partial t} - r K \frac{\partial^5 e}{\partial x^4 \partial t} - g r \frac{\partial^4 e}{\partial x^4} \tag{8}$$

Substituting (7) and (8) in (6), there results

$$r K \frac{\partial^5 e}{\partial x^4 \partial t} + (1 + g r) \frac{\partial^4 e}{\partial x^4} - \frac{L}{l^3} K \frac{\partial^4 e}{\partial x^2 \partial t^2}$$

$$- \left(r C + g \frac{L}{l^3}\right) \frac{\partial^3 e}{\partial x^2 \partial t} - r G \frac{\partial^2 e}{\partial x^2} + \frac{L}{l^3} C \frac{\partial^2 e}{\partial t^2} + \frac{L}{l^3} G \frac{\partial e}{\partial t} = 0 \tag{9}$$

If the losses can be neglected, Equation (9) reduces to

$$\frac{\partial^4 e}{\partial x^4} - \frac{L K}{l^3} \frac{\partial^4 e}{\partial x^2 \partial t^2} + \frac{L}{l^3} C \frac{\partial^2 e}{\partial t^2} = 0 \tag{10}$$

Hereafter it will be convenient to take $l = 1$.

The total current is, from (3)

$$(i_1 + i_2 + i_3) = \left(G + C \frac{\partial}{\partial t}\right) \int e \, d x \tag{11}$$

The solutions to these equations must satisfy

a. The differential equation.
b. The terminal conditions at $x = 0$ and $x = l$.
c. The initial distribution at $t = 0$.
d. The final distribution at $t = \infty$.

If the solution corresponding to a constant sustained potential suddenly applied at $x = l$ can be found, then the solution for any other applied terminal voltage is given by Duhamel's theorem. The usual procedure in solving a partial differential equation is to assume the form of the solution and try it by direct substitution in the differential equation and the boundary conditions. Each tentative trial usually suggests the necessary changes and adjustments in order to meet the complete specifications. Therefore, in order to choose the proper solution from among the infinite number of functions which will satisfy the differential equations, it is necessary to first investigate the boundary conditions.

THE INITIAL DISTRIBUTION

When an infinite rectangular wave is applied at the terminal of the winding, the currents in the capacitances at the first instant are infinite, since the time rate of change of voltage is infinite; whereas the current in the inductive winding is zero, and in the resistances the currents are all finite. Therefore the initial distribution of potential depends only upon the capacitances, Fig. 103B, and can be determined by solving the differential equation for the capacitances alone. Consequently, considering only the capacitances of the circuit, the combination of equations (1) and (3) gives:

$$\frac{\partial^2 e}{\partial x^2} - \frac{C}{K} e = \frac{\partial^2 e}{\partial x^2} - \alpha^2 e = 0 \tag{12}$$

where $\alpha = \sqrt{C/K}$

This equation also follows from the general differential equation (9) upon dividing through by $p^2 = \partial^2 / \partial t^2$ and putting $p = \infty$, according to the procedure in operational calculus.

The solution to (12) is:

$$e = A \epsilon^{\alpha x} + B \epsilon^{-\alpha x} \tag{13}$$

and from (1) the corresponding current is

$$i = K p \frac{\partial e}{\partial x} = K p \alpha (A \epsilon^{\alpha x} - B \epsilon^{-\alpha x}) \tag{14}$$

where $p \rightarrow \infty$, and the initial rush of current is therefore infinite.

Suppose that the winding is grounded at $x = 0$ through a generalized impedance $Z(p)$, and that the voltage applied at $x = 1$ is E. Then

$$\left.\begin{aligned} &\text{at } x = 1,\ e = E = A \epsilon^{\alpha} + B \epsilon^{-\alpha} \\ &\text{at } x = 0,\ e = Z(p)\, i = Z(p)\, p \sqrt{CK}\,(A - B) = A + B \end{aligned}\right\} \tag{15}$$

Herefrom the integration constants are

$$\left.\begin{aligned} A &= \frac{1}{2} \frac{[Z(p)\, p \sqrt{CK} + 1]\, E}{Z(p)\, p \sqrt{CK} \cosh \alpha + \sinh \alpha} \\ B &= \frac{1}{2} \frac{[Z(p)\, p \sqrt{CK} - 1]\, E}{Z(p)\, p \sqrt{CK} \cosh \alpha + \sinh \alpha} \end{aligned}\right\} \tag{16}$$

and the initial distribution therefore is

$$e = E \left. \frac{Z(p)\, p \sqrt{CK} \cosh \alpha x + \sinh \alpha x}{Z(p)\, p \sqrt{CK} \cosh \alpha + \sinh \alpha} \right]_{p=\infty} \tag{17}$$

For a grounded neutral, $Z(p) = 0$, and

$$e = \frac{\sinh \alpha x}{\sinh \alpha} E \tag{18}$$

Equation (18) may be expressed as a half-range sine series

$$e = 2 \sum_{1}^{\infty} \sin s\,\pi\,x \int_{0}^{1} e \sin s\,\pi\,x\,d\,x$$

$$= E \sum_{1}^{\infty} \frac{-\,2\,s\,\pi \cos s\,\pi}{\alpha^2 + s^2\,\pi^2} \sin s\,\pi\,x \tag{19}$$

For an isolated neutral, $Z\,(p) = \infty$, and

$$e = \frac{\cosh \alpha\,x}{\cosh \alpha} E \tag{20}$$

For a capacitance C_0 in the neutral, $Z\,(p) = 1/p\,C_0$, and

$$e = \frac{\sqrt{C\,K} \cosh \alpha\,x + C_0 \sinh \alpha\,x}{\sqrt{C\,K} \cosh \alpha + C_0 \sinh \alpha} E \tag{21}$$

For an inductance L_0 in the neutral, $Z\,(p) = p\,L_0$, and

$$e = \left.\frac{L\,p^2\,\sqrt{C\,K} \cosh \alpha\,x + \sinh \alpha\,x}{L\,p^2\,\sqrt{C\,K} \cosh \alpha + \sinh \alpha}\right]_{p=\infty}$$

$$= \frac{\cosh \alpha\,x}{\cosh \alpha} E \tag{22}$$

A few representative values for $[Z\,(\infty)\ \infty\ \sqrt{C\,K}]$ are given in the following table:

Neutral Impedance	$Z\,(\infty)$	$Z\,(\infty)\ \infty\ \sqrt{C\,K}$
Directly grounded	0	0
Isolated	∞	∞
Resistance R_0	R_0	∞
Inductance L_0	$p\,L_0$	∞
Capacitance C_0	$1/p\,C_0$	$\sqrt{C\,K}/C_0$
L_0 and C_0 in series	$p\,L_0 + 1/p\,C_0$	∞
L_0 and C_0 in parallel	$p\,L_0/(1 + p^2\,L_0\,C_0)$	$\sqrt{C\,K}/C_0$
R_0 and C_0 in parallel	$R_0/(1 + p\,R_0\,C_0)$	$\sqrt{C\,K}/C_0$
R_0 and L_0 in parallel	$R_0\,L_0\,p/(R_0 + p\,L_0)$	∞

It is evident that only an uninterrupted capacitance from neutral to ground, or a directly grounded neutral, can change the initial distribution from that corresponding to an isolated neutral. Curves for the initial distribution are given in Fig. 104 for different values of α and A, and in Fig. 105 the distributions for a grounded neutral are plotted to a larger scale. In an ordinary transformer $5 < \alpha < 30$. Inspection of these distribution curves shows that for values of α in this range there is very little difference, regardless of the neutral connection. These curves also show that the distribution becomes more nearly linear as α decreases, that is, as $\sqrt{C/K}$ decreases. It is thus seen that the distortion of the initial distribution is caused by the capacitance C from winding to ground, and can be improved either by increasing K or decreasing C. This possibility is of primary importance in connection with the schemes of electrostatic shielding of transformers and is discussed in detail in a subsequent chapter.

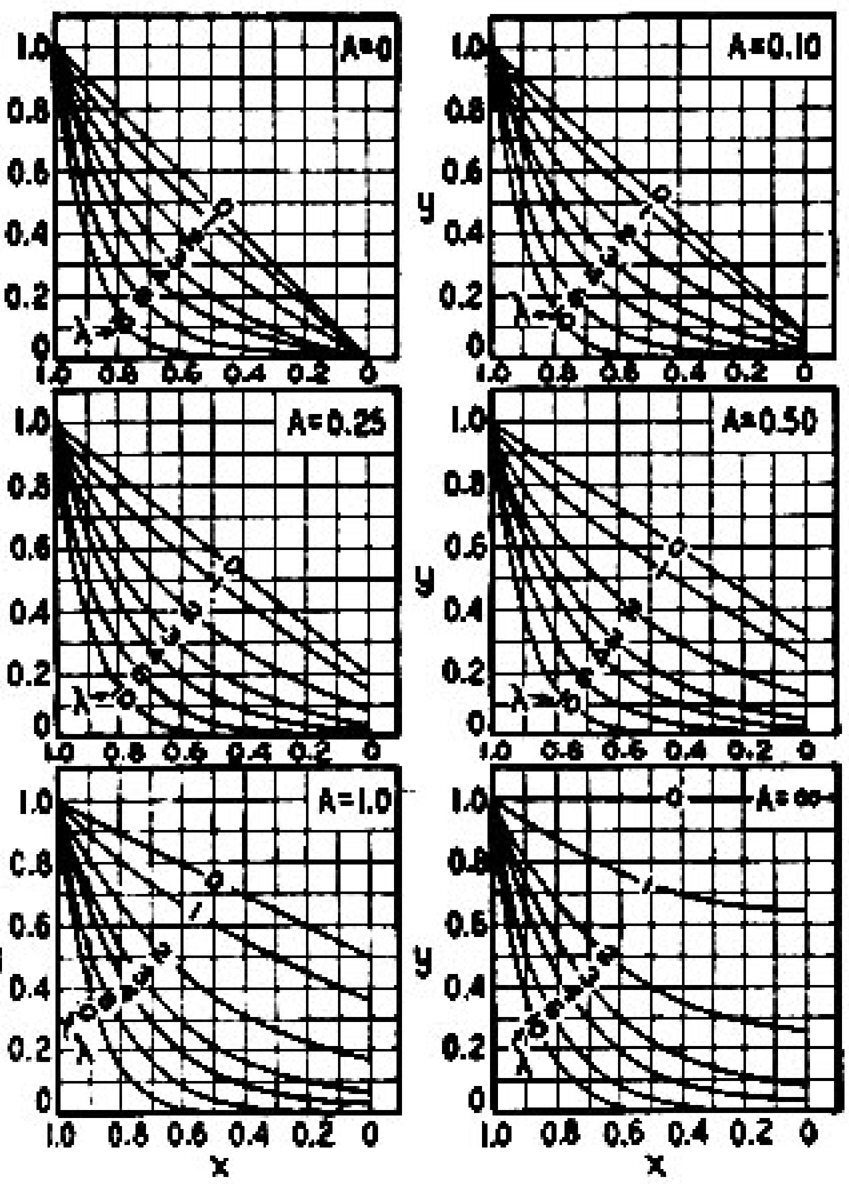


FIG. 104.—Initial and Final Distribution Factor

$$\frac{e}{E} = \frac{A \cosh \lambda x + \sinh \lambda x}{A \cosh \lambda + \sinh \lambda}$$

$\lambda = \alpha$ for initial distribution

$\lambda = \beta$ for final distribution

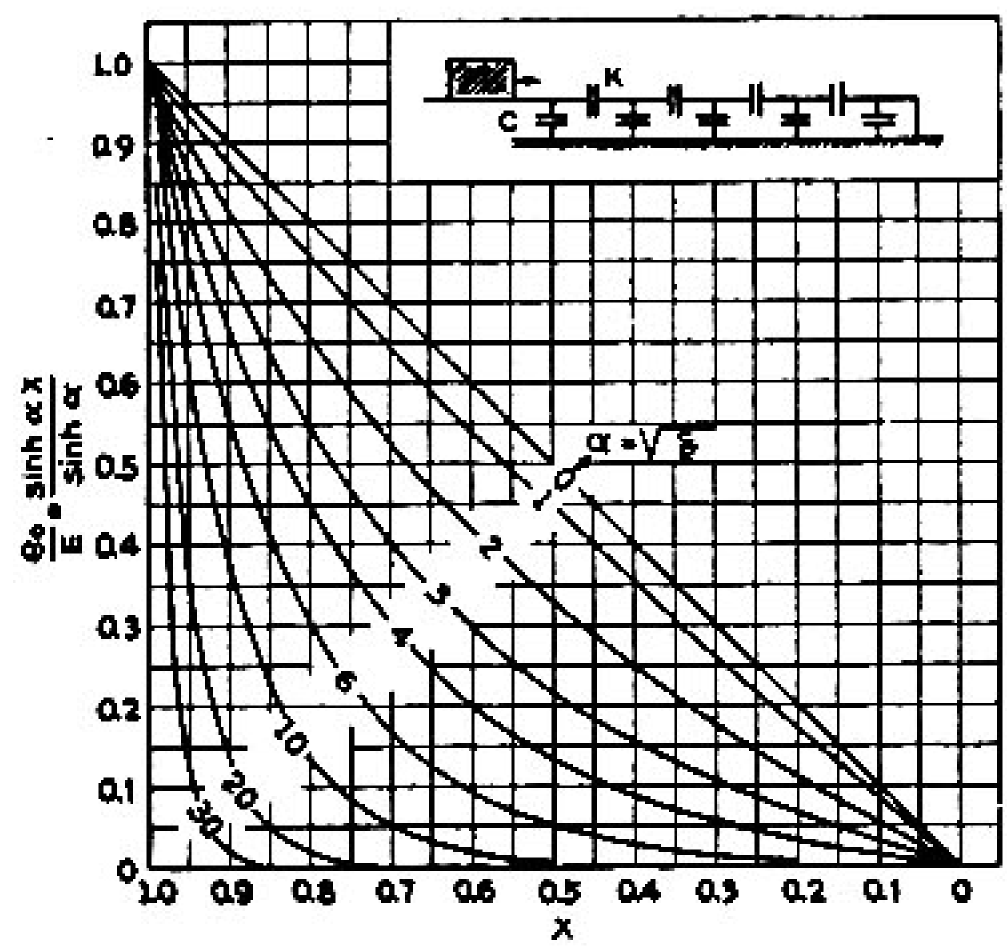


FIG. 105.—Initial Distributions for Grounded Neutral

THE FINAL DISTRIBUTION

After a transient incident to the application of an infinite rectangular wave at the line terminals has died out (theoretically at $t = \infty$) the residual distribution is d-c. The capacitance elements then act as open circuits and the inductance elements as short circuits. The

The normal losses of a transformer are insufficient to exercise much influence on the character of the oscillation, beyond a decrement of the order of 20 per cent per half cycle of fundamental frequency. However, these normal losses very definitely limit the cumulative voltages which may be built up by resonance between the natural oscillation and applied wave frequencies. By increasing the losses sufficiently it is possible to prohibit all oscillations. The initial distribution then diffuses into the final distribution without oscillation, and dangerous abnormal voltages may be avoided. This idea has been applied to current limiting reactors. See Fig. 50.

Equation (43) may be written in several alternative forms as follows

$$e = x\,E + E\sum_{1}^{\infty} A_s \sin s\,\pi\,x \cdot \cos \omega_s t \tag{44a}$$

$$= x\,E + E\sum_{1}^{\infty} \frac{A_s}{2}\left[\sin\,(s\,\pi\,x + \omega_s t) + \sin\,(s\pi x - \omega_s t)\right] \tag{44b}$$

$$= E\sum_{1}^{\infty}\left[A_s \cos\,(s\,\pi - \omega_s t) - \frac{2}{s\pi}\right] \cos s\,\pi \cdot \sin s\pi x \tag{44c}$$

In the case of Fig. 109*A* this last expansion becomes

$$= \sum_{1}^{\infty} \frac{2\,E}{s\pi}\left[\cos s\,\pi\left(1 - \frac{t}{\sqrt{L\,C}}\right) - \cos s\,\pi\right] \sin s\pi x \tag{44d}$$

There are thus three points of view regarding the internal oscillations of distributed circuits of this nature.

Equations	Point of View
44*a*	(Fixed distribution) + (harmonic standing waves)
44*b*	(Fixed distribution) + (pairs of harmonic traveling waves)
44*c*	Simple reflecting traveling wave, in case of Fig. 109*A*.

The *amplitude factors*

$$A_s = \frac{2\,\alpha^2 \cos s\,\pi}{s\,\pi\,(\alpha^2 + s^2\,\pi^2)} \tag{45}$$

have been plotted in Fig. 106. For values of $\alpha > 10$ there is not much change in the envelope of oscillations.

The *ratio of harmonic frequencies* is

$$\frac{f_s}{f_1} = \frac{\omega_s}{\omega_1} = \frac{s^2\,\pi^2}{\sqrt{L(C + K\,s^2\,\pi^2)}}\;\frac{\sqrt{L(C + K\,\pi^2)}}{\pi^2}$$

$$= s^2\sqrt{\frac{C + K\,\pi^2}{C + K\,s^2\pi^2}} = s^2\sqrt{\frac{\alpha^2 + \pi^2}{\alpha^2 + s^2\,\pi^2}} \tag{46}$$

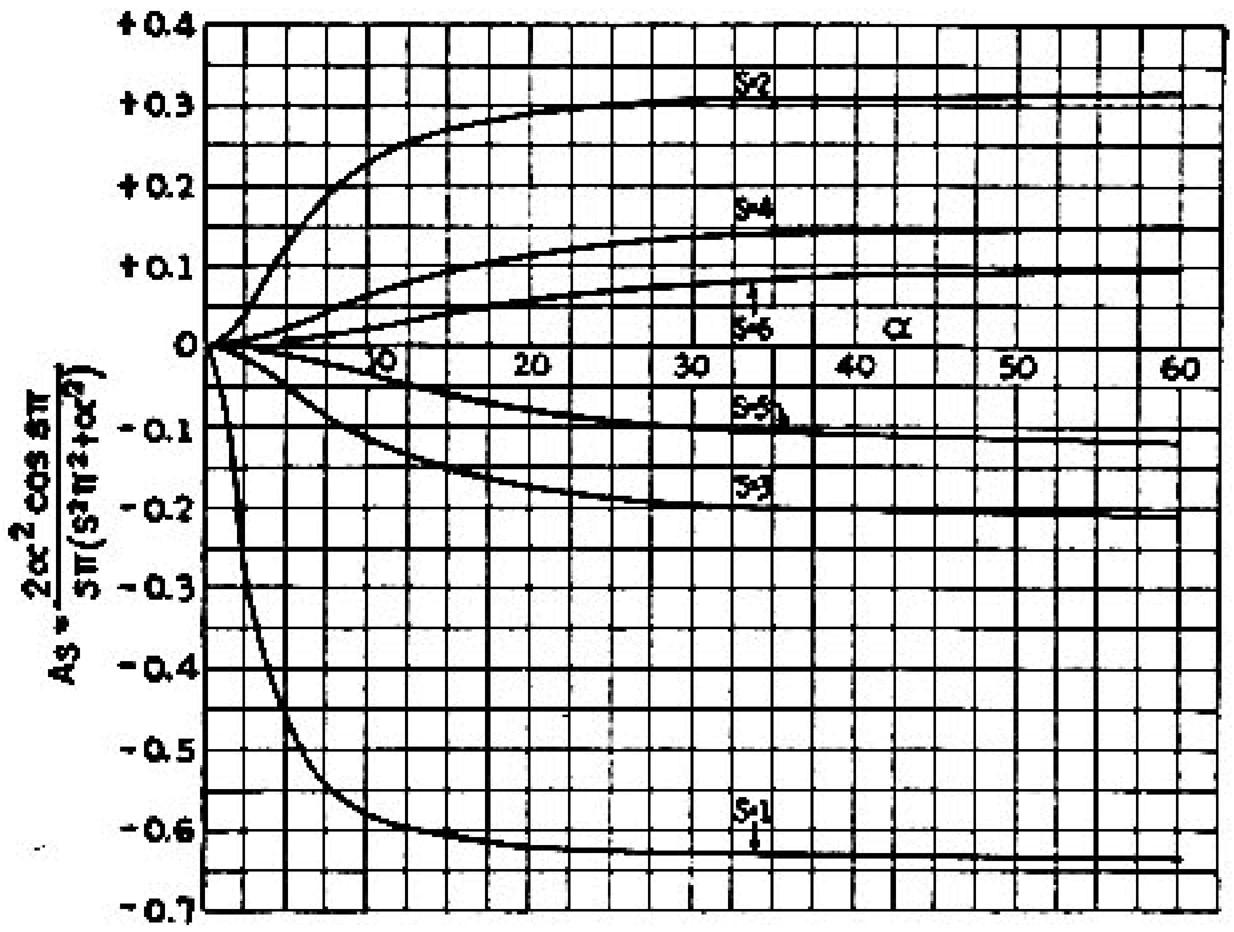


FIG. 106.—Amplitudes of Natural Frequency Oscillations in Transformers for Infinite Rectangular Waves

This ratio has been plotted in Fig. 107.

$$\left.\begin{aligned} \frac{f_s}{f_1} &\cong s^2 \quad \text{for the low harmonics} \\ \frac{f_s}{f_1} &\cong s \quad \text{for the high harmonics} \end{aligned}\right\} \tag{47}$$

The *decrement factors*, from (35), are

$$\gamma_s = \frac{r\,K\,s^4\,\pi^4 + (r\,C + g\,L)\,s^2\,\pi^2 + L\,G}{2\,L\,(C + K\,s^2\,\pi^2)}$$

$$= \frac{r\,K(C/K + s^2\,\pi^2)\,s^2\,\pi^2 + g\,L\,(G/g + s^2\,\pi^2)}{2\,L\,K\,(C/K + s^2\,\pi^2)} \tag{48}$$

Now the conductances G and g depend upon the same geometric factors as the capacitances C and K respectively, so that to a good approximation

$$\frac{G}{g} \cong \frac{C}{K} = \alpha^2 \tag{49}$$

and hereby (48) reduces to

$$\gamma_s = \frac{r}{2L}\left(s^2 \pi^2 + \frac{g L}{r K}\right) = \frac{r}{2L}(s^2 \pi^2 + \sigma) \tag{50}$$

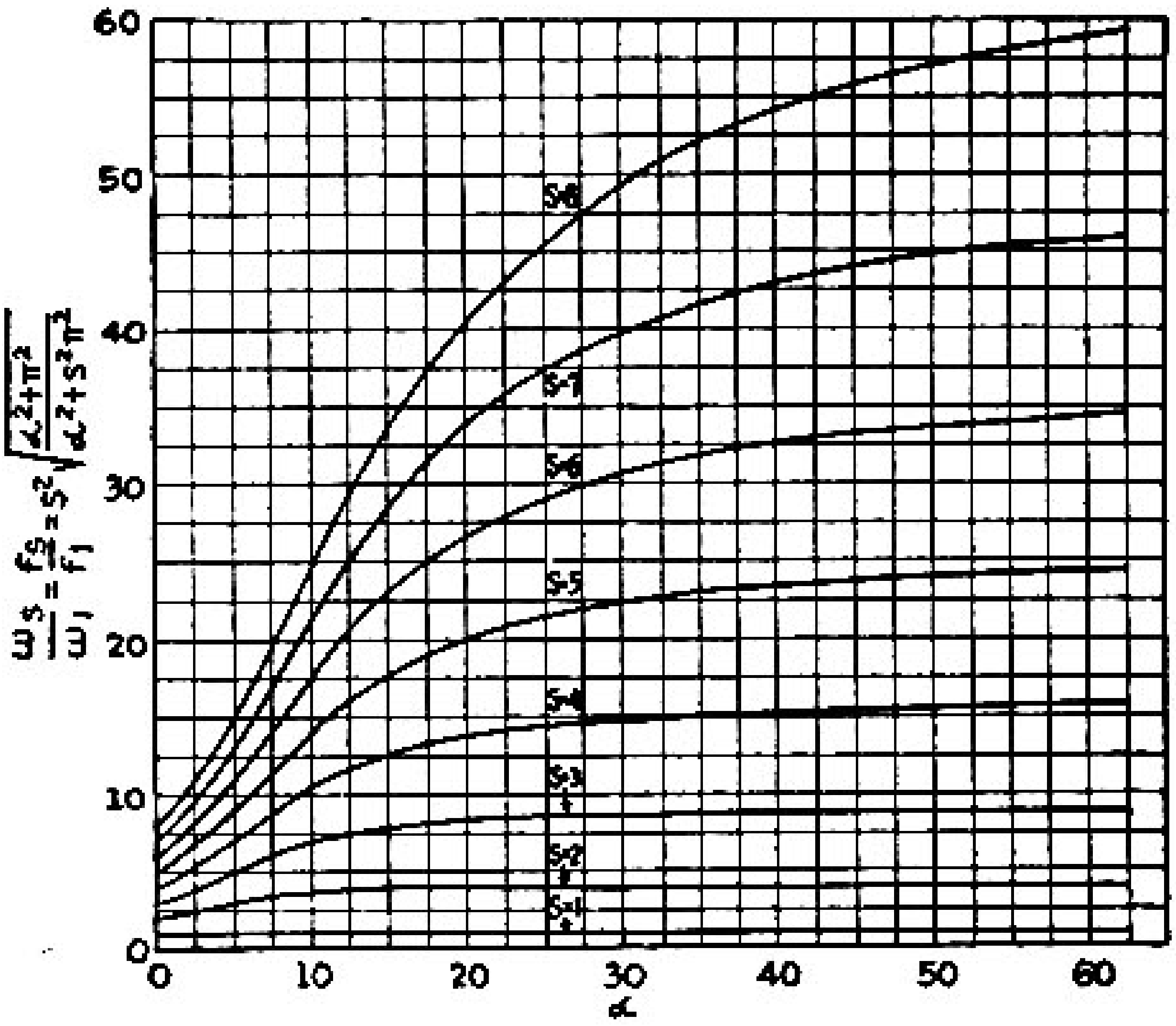


FIG. 107.—Ratio of the Harmonic Natural Frequencies

Therefore, the *ratio of decrement factors*, Fig. 108, is

$$\frac{\gamma_s}{\gamma_1} = \frac{s^2 \pi^2 + \sigma}{\pi^2 + \sigma} \tag{51}$$

where $\sigma = g L / r K$ depends upon four constants: g, L, r, K, none of which is easy to find. It is therefore more feasible to regard σ as an empirical factor which can be obtained from tests by comparing the decrement factors of any two harmonics. It is seen that the decrement factors increase considerably with the order of the harmonic s, about as the square for the higher harmonics. Consequently the higher harmonics are wiped out before the fundamental. Nevertheless, the higher harmonics are important at the neutral end where they pile up; and they also may cause excessive gradients along the stack.

The *effective capacitance* at the line end is defined as

$$C_{eff} = \left.\frac{i}{p E}\right] \text{at } x = 1 \text{ and } t = 0 \tag{52}$$

Then for a grounded neutral, there is, by (14) and (18)

$$C_{eff} = K\,\alpha\,\frac{\cosh\,\alpha}{\sinh\,\alpha} = \sqrt{C\,K}\,\coth\,\alpha \cong \sqrt{C\,K} \tag{53}$$

The *surge impedance* of a harmonic oscillation is defined as

$$z_s = \frac{e_s}{i_s} \tag{54}$$

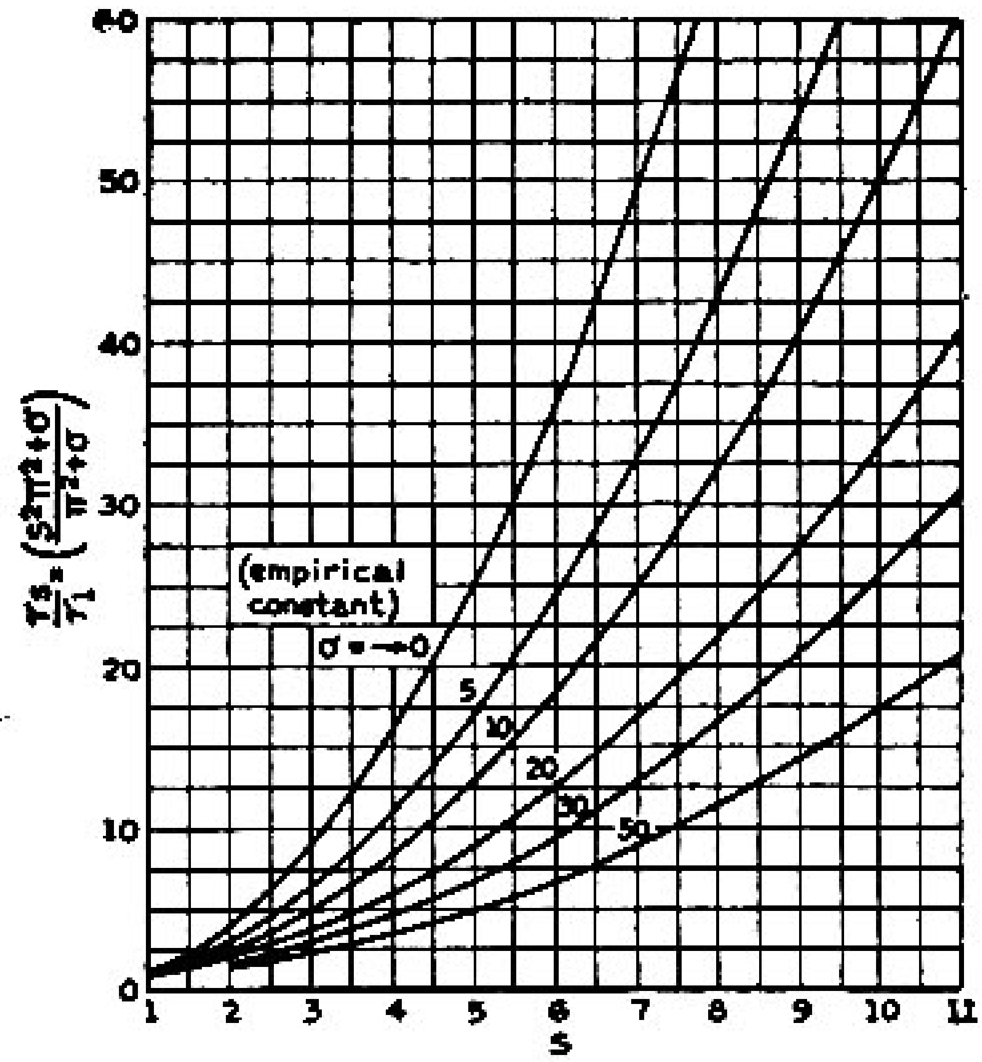


FIG. 108.—Ratio of Decrement Factors of the Natural Oscillations

where e_s is the harmonic voltage and i_s is the corresponding harmonic current which may be calculated by (11). Thus in the case of a grounded neutral

$$e_s = A_s \sin s\,\pi\,x \cos \omega_s t$$

$$i_s = \frac{A_s\,C\,\omega_s}{s\,\pi} \cos s\,\pi\,x \cdot \sin \omega_s t$$

$$z_s = \frac{s\,\pi}{\omega_s\,C} \tag{55}$$

The *velocity of propagation* of a harmonic wave, from (44*b*), is (grounded neutral)

$$v_s = \frac{\omega_s}{s\,\pi} \tag{56}$$

In the appendix to this chapter there are given the derivations for a circuit containing C, K, and L, when the inductance is a pure self-inductance and is not complicated by partial interlinkages. A comparison of this circuit, Fig. 109C, with that previously discussed, Fig. 109D, shows the effects of the partial interlinkages, or mutual inductance between parts of the same winding. Furthermore, by deleting the series capacitance K from the equations its influence can be segregated. The idealized circuits of Fig. 109 represent the range in circuit parameters under consideration, a wavy line indicating pure self-inductance, and a coiled line indicating the presence of mutual inductance between elements of the winding. In the following

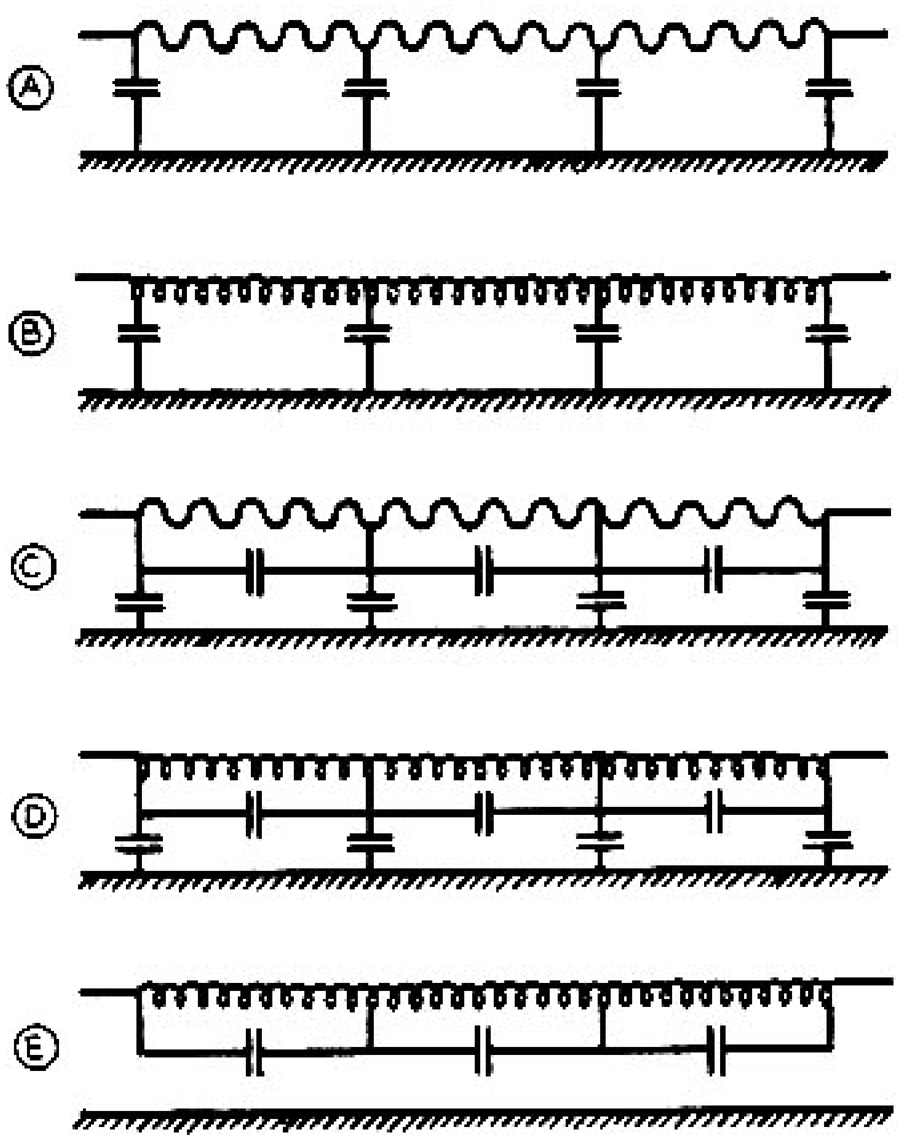


FIG. 109.—Circuits Having Internal Oscillations

Circuit	(L, C)	(M, C)	(L, C, K)	(M, C, K)
A_s	$\frac{2\cos s\pi}{s\pi}$	$\frac{2\cos s\pi}{s\pi}$	$\frac{2\alpha^2\cos s\pi}{s\pi(s^2\pi^2+\alpha^2)}$	$\frac{2\alpha^2\cos s\pi}{s\pi(s^2\pi^2+\alpha^2)}$
ω_s	$\frac{s\pi}{\sqrt{LC}}$	$\frac{s^2\pi^2}{\sqrt{MC}}$	$\frac{s\pi}{\sqrt{L(Ks^2\pi^2+C)}}$	$\frac{s^2\pi^2}{\sqrt{M(Ks^2\pi^2+C)}}$
v_s	$\frac{1}{\sqrt{LC}}$	$\frac{s\pi}{\sqrt{MC}}$	$\frac{1}{\sqrt{L(Ks^2\pi^2+C)}}$	$\frac{s\pi}{\sqrt{M(Ks^2\pi^2+C)}}$
z_s	$\sqrt{\frac{L}{C}}$	$\frac{1}{s\pi}\sqrt{\frac{M}{C}}$	$\sqrt{\frac{L}{C}}\sqrt{\frac{s^2\pi^2}{\alpha^2}+1}$	$\sqrt{\frac{M}{C}}\sqrt{\frac{1}{\alpha^2}+\frac{1}{s^2\pi^2}}$
e $t=0$	0	0	$\frac{\sinh\alpha x}{\sinh\alpha}$	$\frac{\sinh\alpha x}{\sinh\alpha}$
e $t=\infty$	x	x	x	x

table the amplitudes of oscillation (A_s), angular velocities (ω_s), linear velocities of harmonic waves ($\omega_s/s\pi$), surge impedance (z_s) of harmonic waves, initial and final distributions are tabulated. In this table a coefficient L indicates pure self-inductance, and a coefficient M indicates the presence of partial interlinkages.

Transient Distributions.—Distributions corresponding to the four circuits of the above table are plotted in Fig. 110. These curves show the transient voltage distributions along the winding at different instants of time.

If the circuit has only uniformly distributed self-inductance and capacitance to ground (the ideal transmission line) the transient is a

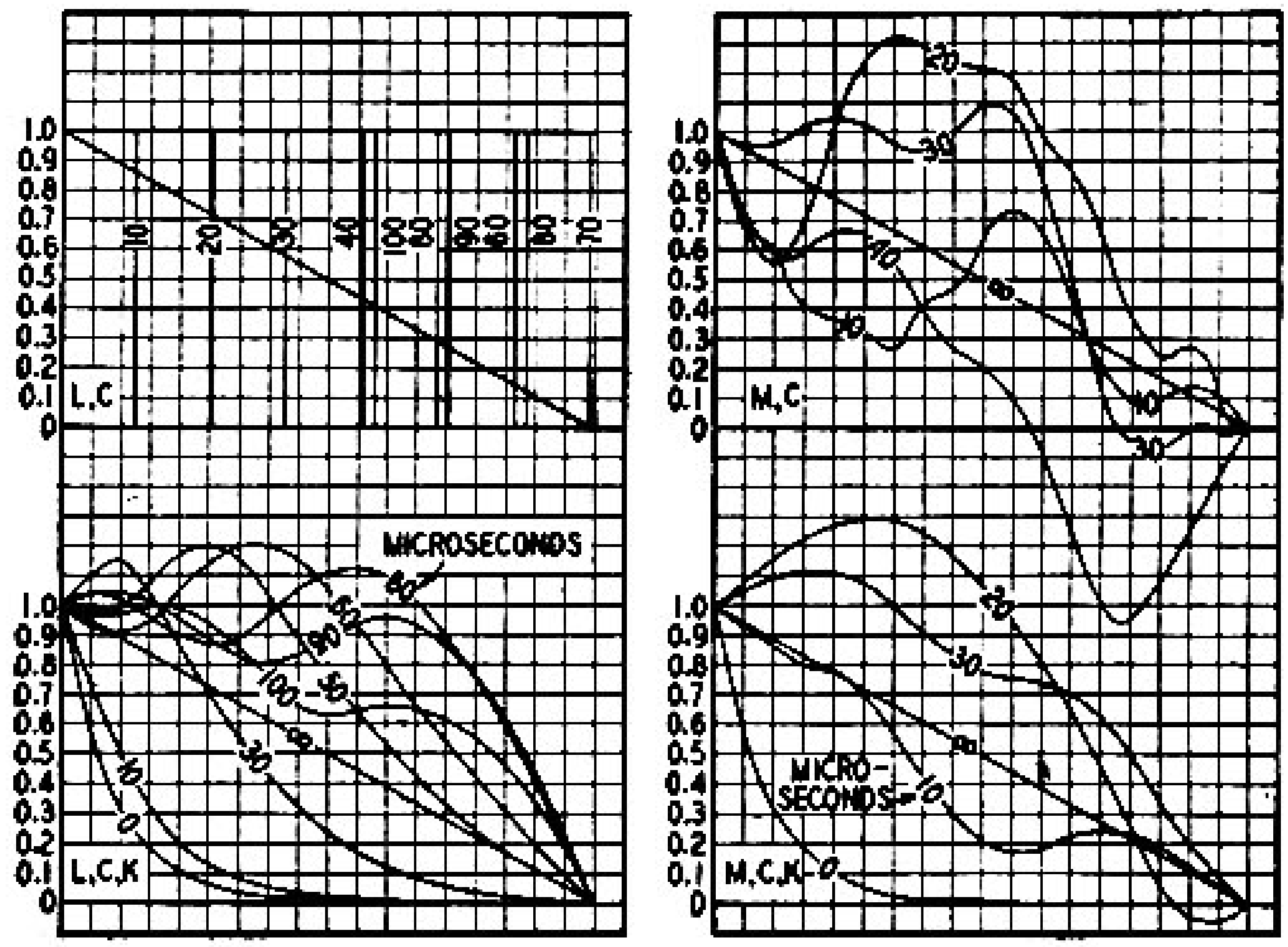


FIG. 110.—Internal Distributions

simple traveling wave which runs back and forth between the line terminal and the neutral at a uniform rate.

The presence of either series capacitance or mutual inductance between parts of the winding introduces distortion, but the mutual inductance is the greater offender in this respect. In general:

a. Series capacitance tends to decrease the amplitudes of oscillation; decrease the frequencies of oscillation, especially of the higher harmonics; decrease the velocity of harmonic waves; increase the surge impedances of harmonics; and better the initial distribution.

b. Mutual inductance tends to increase the frequencies of oscillation; increase the velocities of propagation; and decrease the surge impedances. It has no effect on either the initial distribution or the amplitudes of oscillation.

Inspection of Fig. 110 shows that voltages may occur inside the winding which exceed the applied wave by 30 or 40 per cent. Moreover, gradients far in excess of those corresponding to uniform distribution are experienced at different instants all along the winding. It should be noticed, however, that the initial distribution at the line end gives the maximum gradient that will ever occur anywhere in the winding, for the gradient is

$$\frac{\partial e}{\partial x} = E + \sum_{1}^{\infty} s\,\pi\,A_s \cos s\,\pi\,x \cdot \cos \omega_s t \tag{57}$$

and at $t = 0$ and $x = 1$ all harmonics are in phase (since the sign of A_s is fixed by $\cos s\,\pi$). The second highest gradient occurs at a later time at the neutral.

Potential Difference between Points of the Winding.—The potential difference between any two points x_1 and x_2 of the winding is

$$\begin{aligned} e_1 - e_2 &= (x_1 - x_2)\,E + E\sum_{1}^{\infty} A_s\,(\sin s\pi x_1 - \sin s\pi x_2)\cos \omega_s t \\ &= (x_1 - x_2)E + E\sum_{1}^{\infty} 2\,A_s \sin\frac{s\pi(x_1 - x_2)}{2}\cos\frac{s\pi(x_1 + x_2)}{2}\cos \omega_s t \end{aligned} \tag{58}$$

Consequently the harmonic voltage between any two points vanishes if

$$\left.\begin{aligned} s\,(x_1 + x_2) &= (2\,n - 1) = \text{an odd integer} \\ s\,(x_1 - x_2) &= 2\,n \qquad\quad\;\; = \text{an even integer} \end{aligned}\right\} \tag{59}$$

For example, if $x_1 = 2/3$ and $x_2 = 1/3$, then all the odd harmonics, and all the even harmonics which are multiples of 6, vanish. Thus the only harmonic voltages between these points are the second, fourth, eighth, tenth, fourteenth, etc. Between points equal distances from the ends, $x_2 = (1 - x_1)$, there can exist no odd harmonics, for then

$$s\,(x_1 + x_2) = s \tag{60}$$

Energy in the Oscillations.—At the initial instant the energy of oscillations resides in the capacitances. The harmonic voltages are

$$e = \sum_{1}^{\infty} A_s\,\epsilon^{-\gamma_s t}\cos \omega_s t \sin s\,\pi\,x \tag{61}$$

and at $t = 0$ the energy is

$$W = \frac{E^2}{2}\int_0^1 \left[C\, e^2 + K\left(\frac{\partial e}{\partial x}\right)^2\right] d\,x \tag{62}$$

$$= \frac{E^2}{4}\sum_1^\infty A_s^2\,(C + s^2\,\pi^2\,K)$$

$$= \frac{CE^2}{4}\sum_1^\infty A_s^2\left(1 + \frac{s^2\,\pi^2}{\alpha^2}\right) \tag{63}$$

An alternative expression is given by substituting in (62)

$$e = (e_{t=0} - e_{t=\infty}) = E\left(\frac{\sinh\,\alpha\,x}{\sinh\,\alpha} - x\right) \tag{64}$$

and then

$$W = \frac{CE^2}{6}\left(1 + \frac{3}{\alpha^2} - \frac{3}{\alpha}\coth\,\alpha\right)$$

$$= \frac{CE^2}{6} + \frac{KE^2}{2} - \frac{(\sqrt{C\,K}\coth\,\alpha)\,E^2}{2}$$

$$= \left(\frac{CE^2}{6} + \frac{KE^2}{2}\right) - C_{\text{eff}}\,\frac{E^2}{2} \tag{65}$$

= (final electrostatic energy) − (initial electrostatic energy)

where C_{eff} is the effective terminal capacitance of the transformer at the first instant, as given by (53). In ordinary transformers this energy is of the order of 200 to 500 joules for $E = 1{,}000{,}000$.

In addition to the energy of oscillation there must be supplied to the winding the necessary energy to maintain the terminal voltage E. The rate at which this energy must be supplied depends upon the inductance and resistance of the winding and the connected impedances of both the primary and secondary. If the losses and surge impedances are zero, then the electromagnetic component of current flowing into the winding of inductance L' is

$$I = \frac{E\,t}{L'}$$

and the corresponding energy stored in the electromagnetic field is

$$W_{em} = \frac{L'\,I^2}{2} = \frac{E^2\,t^2}{2\,L'}$$

thus increasing as the square of the time. But if the series resistance of the winding is r, then the supply of energy necessary to maintain the terminal voltage is ultimately at a rate of E^2/r. More generally, considering the connected impedances and self and mutual inductances of the windings, the rate of energy supply must be computed from Equations (49) and (50) of Chapter XII. These equations are considered in detail under "Secondary Terminal Transients" in Chapter XV.

SOLUTION FOR ISOLATED NEUTRAL

The analysis for an isolated neutral proceeds along exactly the same lines as that for the grounded neutral. Equations (20) and

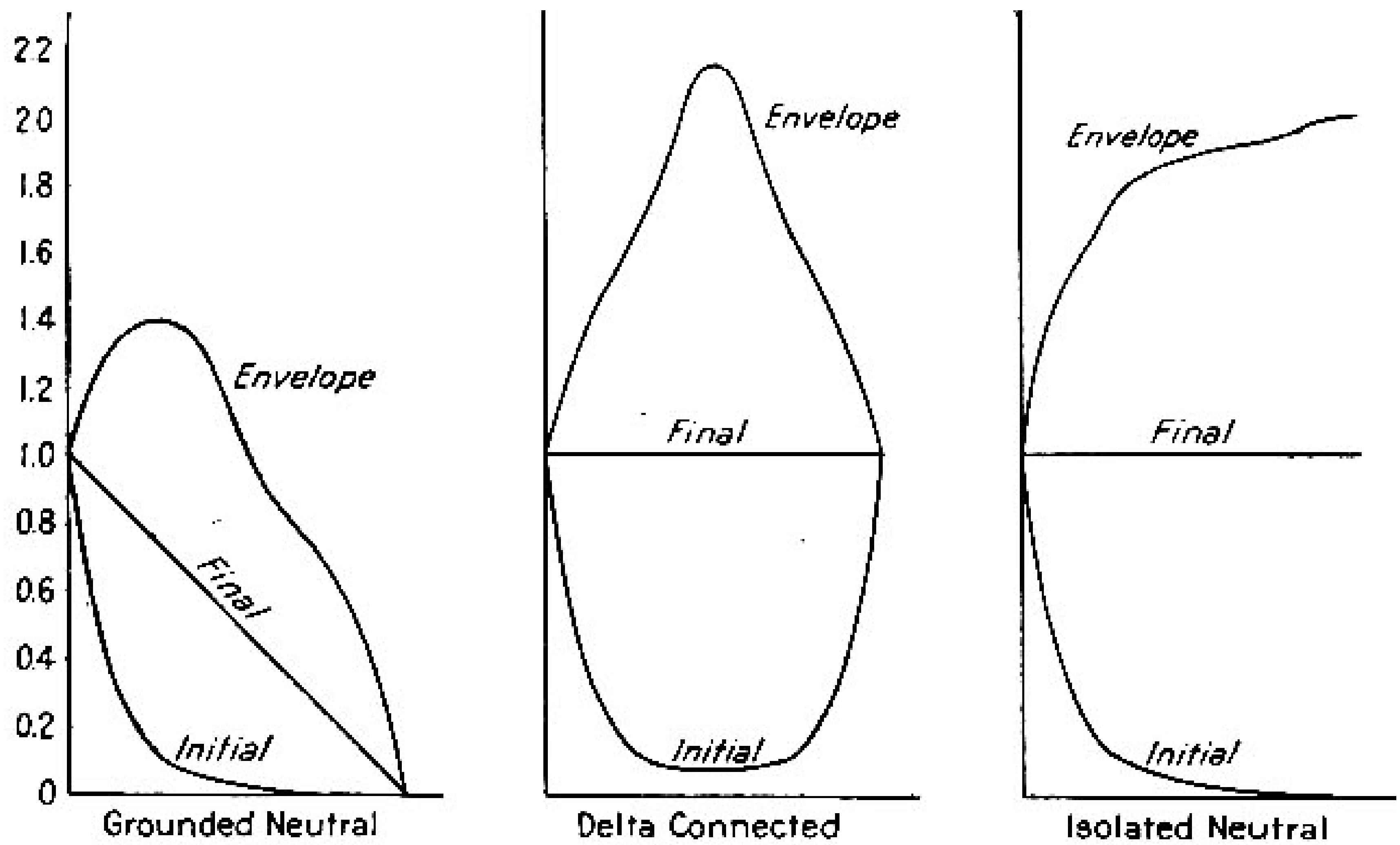


FIG. 111.—Typical Initial and Final Distributions and Envelopes of Oscillation

(30) are employed for the initial and final distributions respectively, and the Fourier expansion of these distributions is made as a quarter-range *cosine* series of odd harmonics,

$$\cos\,(2\,s - 1)\,\frac{\pi\,x}{2} \tag{66}$$

The solution for no losses and an infinite rectangular applied wave is

$$e = E + E\sum_{1}^{\infty} B_s \cos\frac{s\,\pi\,x}{2}\cos\Omega_s t \tag{67}$$

where

$$B_s = \frac{-16\,\alpha^2 \sin s\,\pi/2}{s\,\pi\,(s^2\,\pi^2 + 4\,\alpha^2)} \tag{68}$$

$$\Omega_s = \frac{s^2\,\pi^2/4}{\sqrt{L(C + K\,s^2\,\pi^2/4)}} \tag{69}$$

The characteristic parameters for the circuits of Fig. 109 are tabulated below:

Circuit	(L, C)	(M, C)	(L, C, K)	(M, C, K)
B_s	$\dfrac{4 \sin s\pi/2}{s\pi}$	$\dfrac{4 \sin s\pi/2}{s\pi}$	$\dfrac{16\alpha^2 \sin s\pi/2}{s\pi(s^2\pi^2 + 4\alpha^2)}$	$\dfrac{16\alpha^2 \sin s\pi/2}{s\pi(s^2\pi^2 + 4\alpha^2)}$
Ω_s	$\dfrac{s\pi/2}{\sqrt{LC}}$	$\dfrac{s^2\pi^2/4}{\sqrt{MC}}$	$\dfrac{s\pi/2}{\sqrt{L(Ks^2\pi^2/4 + C)}}$	$\dfrac{s^2\pi^2/4}{\sqrt{M(Ks^2\pi^2/4 + C)}}$
V_s	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{s\pi/2}{\sqrt{MC}}$	$\dfrac{1}{\sqrt{L(Ks^2\pi^2/4 + C)}}$	$\dfrac{s\pi/2}{\sqrt{M(Ks^2\pi^2/4 + C)}}$
Z_s	$\sqrt{\dfrac{L}{C}}$	$\dfrac{2}{s\pi}\sqrt{\dfrac{M}{C}}$	$\sqrt{\dfrac{L}{C}\left(\dfrac{s^2\pi^2}{4\alpha^2} + 1\right)}$	$\sqrt{\dfrac{M}{C}\left(\dfrac{1}{\alpha^2} + \dfrac{4}{s^2\pi^2}\right)}$
e $t = 0$	0	0	$E\dfrac{\cosh \alpha x}{\cosh \alpha}$	$E\dfrac{\cosh \alpha x}{\cosh \alpha}$
e $t = \infty$	E	E	E	E

The energy in the oscillation is

$$W = (C - C_{\text{eff}})\frac{E^2}{2}$$

APPENDIX TO CHAPTER XIII

NEGLIGIBLE MUTUAL INDUCTANCE AND NO LOSSES

The fundamental circuit equations are

$$i_1 = K \frac{\partial^2 e}{\partial x \, \partial t} \tag{1}$$

$$i_4 = C \frac{\partial e}{\partial t} = \frac{\partial}{\partial x}(i_1 + i_2) \tag{2}$$

$$\frac{\partial e}{\partial x} = L \frac{\partial i_2}{\partial t} \tag{3}$$

From these three equations the differential equation is

$$K L \frac{\partial^4 e}{\partial x^2 \, \partial t^2} + \frac{\partial^2 e}{\partial x^2} - L C \frac{\partial^2 e}{\partial t^2} = 0 \tag{4}$$

or substituting $p = \partial / \partial t$ and rewriting in symbolic notation

$$\frac{\partial^2 e}{\partial x^2} - \frac{L C p^2}{K L p^2 + 1} e = 0 \tag{5}$$

The solution to (5) with respect to x is

$$e = A \epsilon^{\sigma x} + B \epsilon^{-\sigma x} \tag{6}$$

where

$$\sigma = \sqrt{\frac{L C p^2}{K L p^2 + 1}} \tag{7}$$

From (2) and (6) the total current in the series path is

$$i = (i_1 + i_2) = \int C \frac{\partial e}{\partial t} d x = \frac{C p}{\sigma}(A \epsilon^{\sigma x} - B \epsilon^{-\sigma x}) \tag{8}$$

The terminal conditions are

$$\left.\begin{aligned} e &= E \text{ at } x = 1 \\ Z(p)\, i &= e \text{ at } x = 0 \end{aligned}\right\} \tag{9}$$

From (6), (8), and (9) the integration constants are

$$\left.\begin{aligned} A &= \frac{C p Z(p) + \sigma}{C p Z(p) \cosh \sigma + \sigma \sinh \sigma} \frac{E}{2} \\ B &= \frac{C p Z(p) - \sigma}{C p Z(p) \cosh \sigma + \sigma \sinh \sigma} \frac{E}{2} \end{aligned}\right\} \tag{10}$$

and substituting (10) in (6) there is

$$e = \frac{C\,p\,Z\,(p)\cosh\sigma x + \sigma\sinh\sigma x}{C\,p\,Z\,(p)\cosh\sigma + \sigma\sinh\sigma}\,E = \frac{Y\,(p)}{H\,(p)} \tag{11}$$

This operational equation may be solved by the Heaviside expansion theorem

$$e = \frac{Y\,(p)}{H\,(p)} = \frac{Y\,(0)}{H\,(0)} + \sum \frac{Y\,(p_s)\,\epsilon^{p_s t}}{p_s\,H'\,(p_s)} \tag{12}$$

where the summation is to include all the roots of $H\,(p) = 0$.

Applying (12) to (11) there is

$$Y\,(p) = C\,p\,Z\,(p)\cosh\sigma x + \sigma\sinh\sigma x \tag{13}$$

$$H\,(p) = C\,p\,Z\,(p)\cosh\sigma + \sigma\sinh\sigma \tag{14}$$

$$\frac{Y\,(0)}{H\,(0)} = \frac{Z\,(0) + p\,L\,x}{Z\,(0) + p\,L}\Bigg/_{p\,=\,0} \tag{15}$$

The roots of (14) are given by the transcendental equation

$$\tanh\sigma = -\frac{C\,p\,Z\,(p)}{\sigma} \tag{16}$$

and from (7)

$$p = \pm\sqrt{\frac{\sigma^2}{L\,(C - \sigma^2 K)}} \tag{17}$$

In the general case, the roots of (16) must be found by methods of approximation.

$$p\,\frac{d\,H\,(p)}{d\,p} = \left(\sinh\sigma + \frac{\sigma}{\cosh\sigma}\right)\left(1 - \frac{\sigma^2}{\alpha^2}\right)\sigma$$

$$+ \,p_s\cosh\sigma\cdot C\,\frac{d}{d\,p}\,[p\,Z\,(p)] \tag{18}$$

This is as far as the generalization can be carried.

If the neutral impedance is a capacitance C_0, then

$$Z\,(p) = \frac{1}{p\,C_0} \tag{19}$$

and the last term of (18) vanishes. Now substituting $\sigma = j\,\zeta$, Equation (16) becomes

$$\frac{C_0}{C}\,\zeta = \cot\zeta = \frac{1}{\zeta} - \frac{\zeta}{3} - \frac{\zeta^3}{45} - \frac{2}{945}\,\zeta^5 - \frac{\zeta^7}{4725} - \dots \tag{20}$$

or

$$\zeta^8 + 10\,\zeta^6 + 105\,\zeta^4 + \left(1575 - 4725\,\frac{C_0}{C}\right)\zeta^2 - 4725 \cong 0$$

from which the most important roots of (16) may be determined by approximations. Suppose that this has been done for a specific ratio C_0/C. Then by (17)

$$p = \pm\frac{j\,\zeta}{\sqrt{L\,(C + \zeta^2 K)}} = \pm\,j\,\omega \tag{21}$$

Substituting (13), (15), (18), and (21) in the expansion formula (12), there results

$$e = E - 2E \sum \frac{C \cos \zeta x - C_0 \zeta \sin \zeta x}{C_0 \left(\sin \zeta + \dfrac{\zeta}{\cos \zeta} \right) \zeta \left(1 + \dfrac{\zeta^2}{\alpha^2} \right)} \cos \omega t \tag{22}$$

where the roots are given by Equation (20).

For a grounded neutral, $Z(p) = 0$, and (16) gives

$$\sinh \sigma = \sigma \prod_1^\infty \left(1 + \frac{\sigma^2}{s^2 \pi^2} \right) = 0 \tag{23}$$

from which

$$\sigma = \pm j \zeta = \pm j s \pi \tag{24}$$

and the solution is

$$e = x E + E \sum_{s=1}^{\infty} \frac{\cos s \pi}{s \pi}$$

$$\frac{2 \alpha^2}{\alpha^2 + s^2 \pi^2} \sin s \pi x \cos \frac{s \pi t}{\sqrt{L(C + s^2 \pi^2 K)}} \tag{25}$$

For an isolated neutral, $Z(p) = \infty$, and (16) gives

$$\cosh \sigma = \prod_1^\infty \left[1 + \frac{4 \sigma^2}{\pi^2 (2s - 1)^2} \right] = 0 \tag{26}$$

from which

$$\sigma = \pm j \zeta = \pm j \frac{(2s - 1)\pi}{2} \tag{27}$$

and the solution is

$$e = E + E \sum_1^\infty \frac{\sin s \pi / 2}{s \pi}$$

$$\frac{16 \alpha^2}{s^2 \pi^2 + 4 \alpha^2} \cos \frac{s \pi x}{2} \cos \frac{s \pi t}{\sqrt{L(4C + K s^2 \pi^2)}} \tag{28}$$

When $K = 0$ and $\alpha = \sqrt{C/K} \to \infty$, this equation reduces to the equation of an open end transmission line, or

$$e = E + 4E \sum_1^\infty \frac{\sin s \pi / 2}{s \pi} \cos \frac{s \pi x}{2} \cos \frac{s \pi t}{2\sqrt{L C}} \tag{29}$$

SUMMARY OF CHAPTER XIII

The idealized circuit of the single-winding theory of transformer oscillations, including the losses, is characterized by a fifth-order partial differential equation which may be solved for either a grounded or isolated neutral. The initial and final distributions may be found for general impedances in the neutral, and both distributions have the same functional form. The general solutions for grounded or isolated neutral are in the form of an infinite series of damped space and time harmonics oscillating about the final distribution as axis of oscillation. The form of the solution shows that the oscillations may be suppressed entirely by making the final and initial distributions coincident—the criterion for electrostatic shielding discussed in a subsequent chapter. The effects of the losses as well as of all the other circuit constants are evaluated in the text and illustrated by curves. Formulas and curves are also given for the amplitudes and frequencies of oscillation, the harmonic decrement factors, the effective capacitance of the transformer, the surge impedances and velocities of propagation of the harmonic waves, and the initial and final distributions, as well as typical envelopes of oscillation for grounded and isolated neutral, and delta-connected transformers. Equation (58) gives the potential difference between points on the winding. Equation (57) specifies the gradient or axial stress along the winding; (63) and (65) are equations for the energy of oscillations.

CHAPTER XIV

WAVES OF ARBITRARY SHAPE APPLIED AT ONE OR BOTH TERMINALS

Effect of Applied Wave Shape.—Equation (41) applies when an infinite rectangular wave is impressed at the primary line terminal. If the applied wave is $E\ (t)$ the corresponding solution may be obtained by Duhamel's theorem, two expressions for which yield

$$e = \frac{\sinh \alpha\, x}{\sinh \alpha} E(t) - \sum_1^\infty \int_0^t A_s\, \epsilon^{-\gamma_s(t-\tau)}[\gamma_s \cos \omega_s\, (t-\tau) + \omega_s \sin \omega_s\, (t-\tau)]\, E(\tau) \sin s\, \pi\, x\, d\tau$$

$$= \frac{\sinh \beta\, x}{\sinh \beta} E(t) + \sum_1^\infty A_s \left\{ E(t) - \int_0^t \epsilon^{-\gamma_s(t-\tau)} [\gamma_s \cos \omega_s\, (t-\tau) + \omega_s \sin \omega_s\, (t-\tau)]\, E(\tau)\, d\tau \right\} \sin s\pi x \quad (70)$$

Either one of these two expressions may be employed. The former reduces directly to the initial distribution (18) at $t = 0$, and the latter reduces directly to the final distribution (29) at $t = \infty$. In subsequent applications only the latter equation will be used. If the losses are zero it becomes

$$e = x\, E(t) + \sum_1^\infty A_s \left\{ E(t) - \int_0^t \omega_s \sin \omega_s\, (t-\tau)\, E(\tau)\, d\tau \right\} \sin s\pi x \quad (71)$$

In any event, it is permissible to take

$$\frac{\sinh \beta\, x}{\sinh\ \beta} \cong x \quad (72)$$

Finite Rectangular Wave, Fig. 112*B*.—For a finite rectangular wave of length L the solution is the same as (44*a*) up to $t = L$, but for greater values of time the solution is obtained by superimposing a negative rectangular wave at $t = L$.

$$E(t) = \begin{cases} E \text{ for } 0 < t < L \\ 0 \text{ for } t > L \end{cases}$$

and

$$e_{(t<L)} = x\,E + E \sum_{1}^{\infty} A_s \sin s\,\pi\,x \cdot \cos \omega_s t \tag{73}$$

$$e_{(t>L)} = 0 + E \sum_{1}^{\infty} A_s \sin s\,\pi\,x\,[\cos \omega_s t - \cos \omega_s (t - L)]$$

$$= 0 - E \sum_{1}^{\infty} A_s \sin s\,\pi\,x \left(2 \sin \frac{\omega_s L}{2}\right) \sin \omega_s \left(t - \frac{L}{2}\right) \tag{74}$$

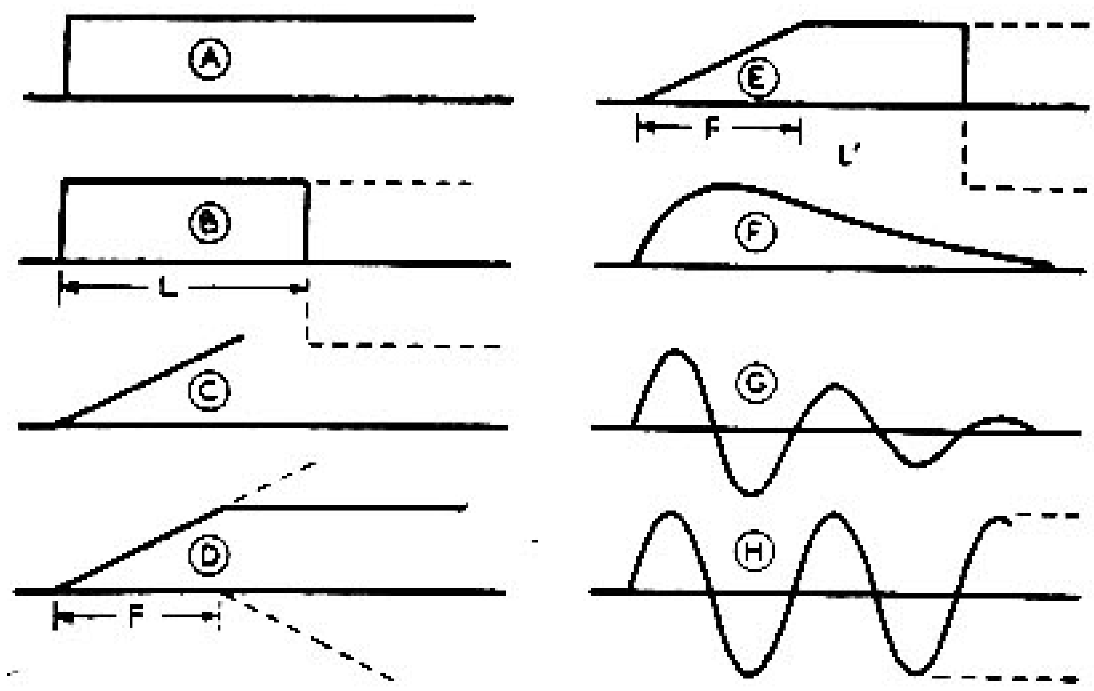


FIG. 112.—Wave Shapes Used in Calculations

Thus at $t = L$ the axis of oscillations shifts from the $x\,E$ line to the zero axis, but the amplitudes of oscillation are multiplied by the factor

$$2 \sin \frac{\omega_s L}{2} \tag{75}$$

This factor has been plotted in Fig. 114. Any harmonic is then a maximum for

$$\frac{\omega_s L}{2} = (2\,n - 1)\,\frac{\pi}{2} \tag{76}$$

where n is any integer. Therefore the critical wave length which doubles the amplitude of oscillation is

$$L = (2\,n - 1)\,\frac{\pi}{\omega_s} = (2\,n - 1)\,\frac{\pi}{2\,\pi\,f_s} = (2\,n - 1)\,\frac{T_s}{2} \tag{77}$$

where $T_s = 1/f_s$ is the natural period of oscillation.

Therefore, if the wave length is an odd multiple of the natural period of any harmonic, that harmonic will be doubled. However, the maximum crest of the envelope of oscillations will not necessarily be increased, because of the shift of the axis of oscillations to the zero line. For example, considering only the axis of oscillation and the harmonic which conforms to (77)

$$e' = E\,(x + A_s \sin s\,\pi\,x \cos \omega_s t) \text{ for } t < L \tag{78}$$

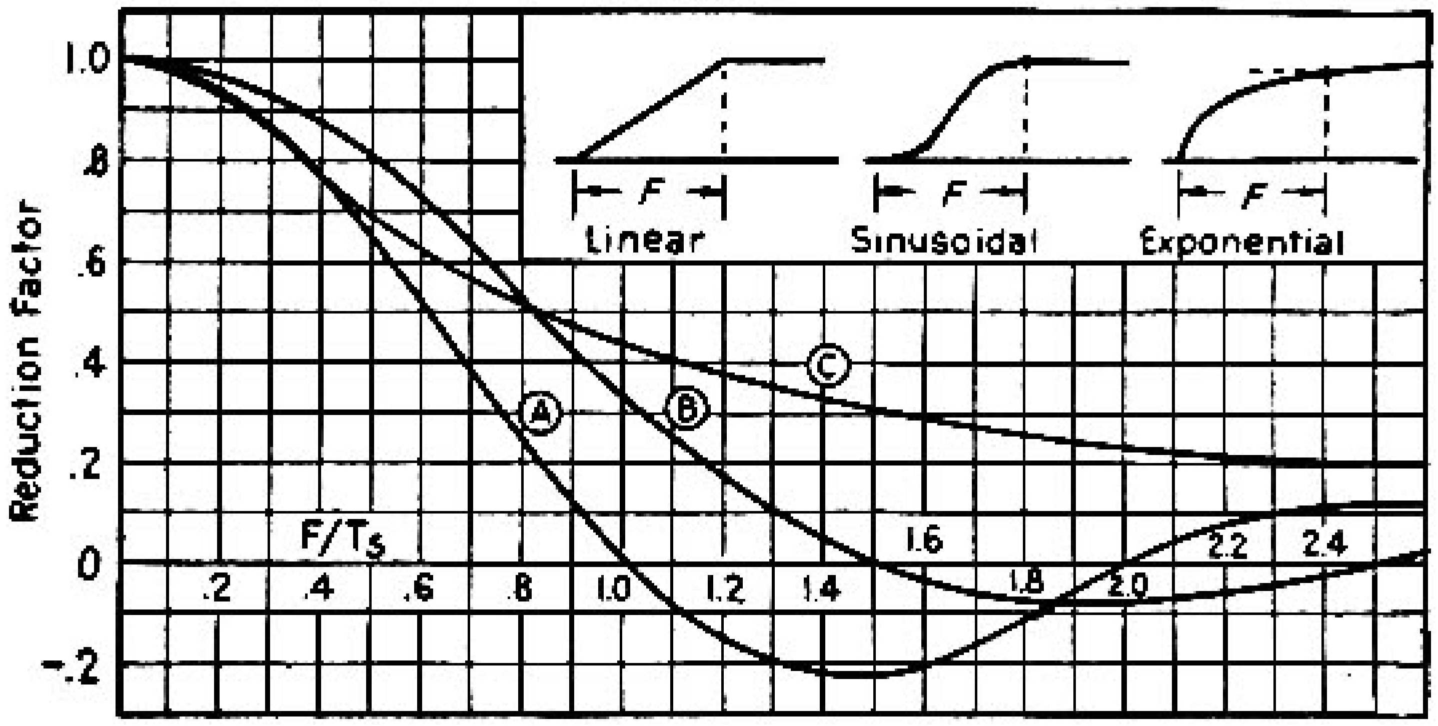


FIG. 113.—Reduction Factors for Wave Fronts

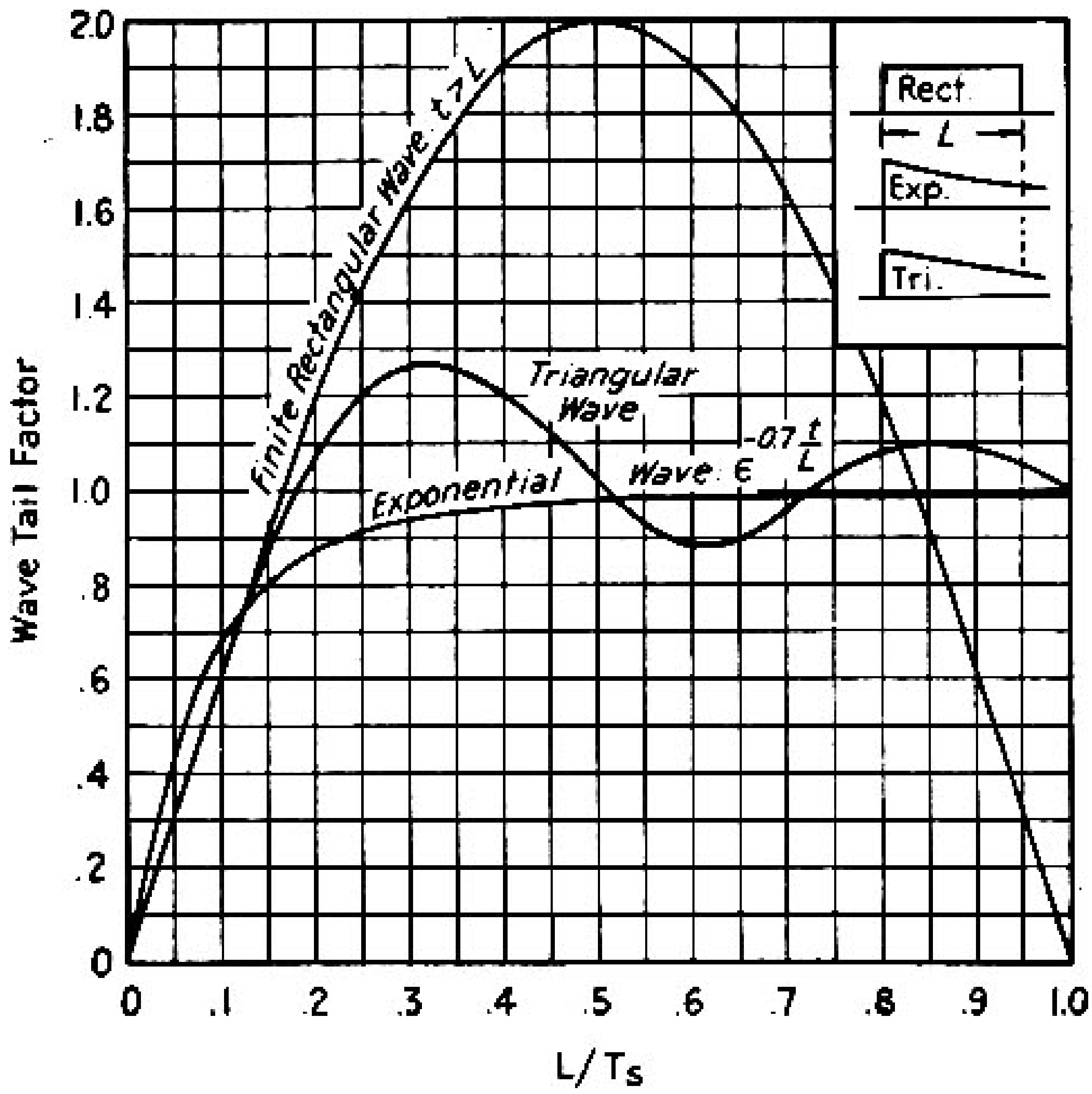


FIG. 114.—Reduction Factors for Wave Tails

This is a maximum for $\cos \omega_s t = \pm 1$ and for

$$\frac{\partial e'}{\partial x} = 0 = (1 \pm s\,\pi\,A_s \cos s\,\pi\,x)$$

Therefore

$$x = \frac{1}{s\,\pi} \cos^{-1}\left(\frac{\pm 1}{s\,\pi\,A_s}\right)$$

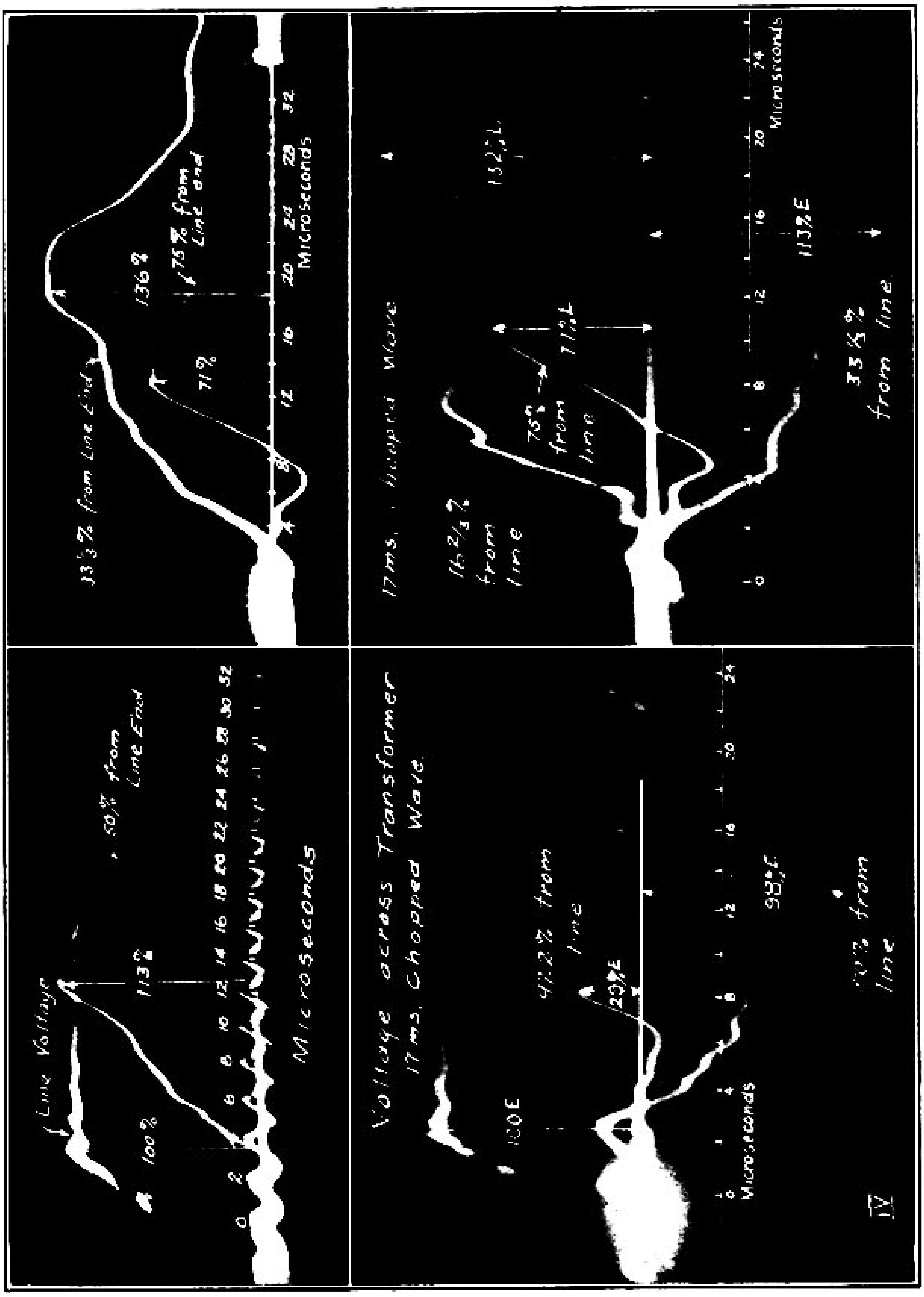


FIG. 115.—Effect of Chopping the Applied Wave on the Tail

Therefore

$$e'_{\max} = E\left[\frac{1}{s\,\pi}\cos^{-1}\left(\frac{\pm 1}{s\,\pi\,A_s}\right) + A_s\sqrt{1 - \frac{1}{(s\,\pi\,A_s)^2}}\right]$$

$$= \frac{E}{s\,\pi}\left[\cos^{-1}\left(\frac{\pm 1}{s\,\pi\,A_s}\right) + \sqrt{s^2\,\pi^2\,A_s^2 - 1}\right] \tag{79}$$

In a grounded-neutral winding, A_s is greatest for the fundamental, but can not exceed $2/\pi$, so that the maximum to which the fundamental can raise the voltage is

$$e'_{\max} = \frac{E}{\pi}\left[\cos^{-1}\left(\frac{-1}{2}\right) + \sqrt{3}\right] = 1.218\,E \text{ for } t < L \tag{80}$$

as compared with

$$e''_{\max} = 2\,A_s\,E = \frac{4}{\pi}\,E = 1.272\,E \text{ for } t > L \tag{81}$$

These relative values are, of course, altered by the contributions of the other harmonics, and A_s is always less than $2\ \pi$.

Linear Front, Fig. 112 *C-D-E*.—Substituting the linear front, Fig. 112*C*,

$$E(t) = a\,t \tag{82}$$

in (71) there results

$$e = a\,t\,x + \sum_1^\infty \frac{a}{\omega_s} A_s \sin s\,\pi\,x \sin \omega_s t$$

and the distribution is directly proportional to the steepness of the wave front.

Herefrom the solution for a wave with a linear front and an infinite tail, Fig. 112*D*, is readily found by superimposing two such waves of opposite sign and displaced by $t = F$. Then

$$\left.\begin{aligned} e_{t<F} &= x\,\frac{E}{F}\,t + E\sum_1^\infty \frac{A_s}{\omega_s\,F} \sin s\,\pi\,x \sin \omega_s t \\ e_{t>F} &= x\,E + E\sum_1^\infty A_s\left[\frac{\sin \omega_s\,F\ 2}{\omega_s\,F\ 2}\right]\cos \omega_s\left(t - \frac{F}{2}\right)\sin s\,\pi\,x \end{aligned}\right\} \tag{83}$$

For $t > F$ the amplitude of any harmonic is reduced by the reduction factor

$$\left[\frac{\sin \omega_s\,F\ 2}{\omega_s\,F\ 2}\right] \tag{84}$$

This factor has been plotted in Fig. 113*A*, and its effect is illustrated by the cathode-ray oscillograms of Fig. 116. It is zero for

$$\frac{\omega_s\,F}{2} = n\,\pi \quad \text{or} \quad F = \frac{2\,n\,\pi}{\omega_s} = n\,T_s \tag{85}$$

Thus if the wave front is a multiple of the natural period of oscillation of a particular harmonic, then that harmonic vanishes. This fact was pointed out by K. K. Palueff in his 1929 paper. It is also evident that, if the fundamental is wiped out by this means, all the harmonics

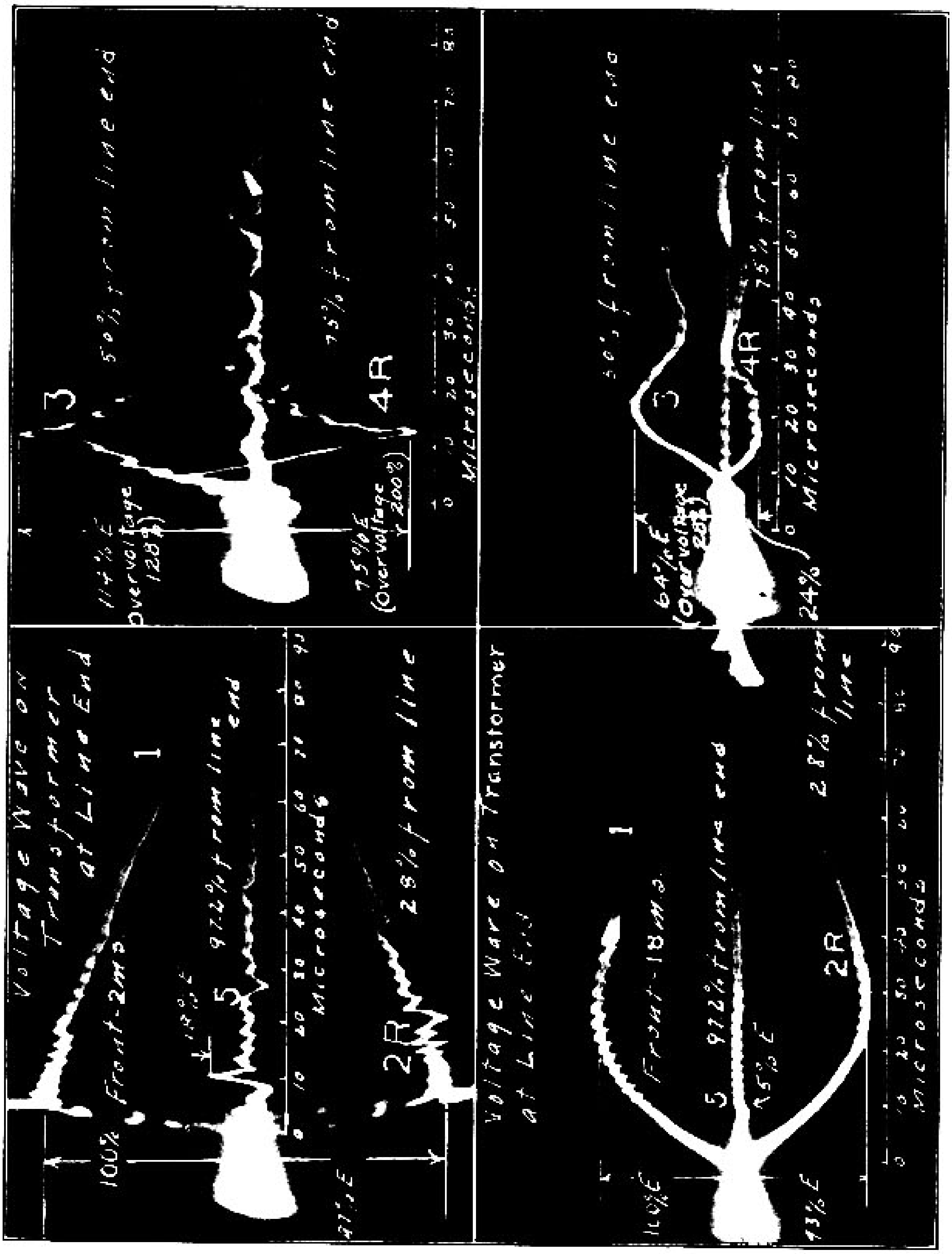


FIG. 116.—Effect of Wave Front
Top: Applied Wave of 3 ms. Front. Bottom: Applied Wave of 18 ms. Front

will also practically vanish. As the wave front increases, the higher harmonics are wiped out first.

When a wave of this type is chopped on the tail, Fig. 112*E*, the subsequent distribution is found by superimposing an infinite rectangular wave of opposite sign, and displaced back from the origin by an amount $t = L$. The solution then is

$$e_{t>L} = E\sum_{1}^{\infty} A_s \left[-\cos \omega_s (t - L) + \left(\frac{\sin \omega_s F\,2}{\omega_s F\,2}\right) \cos \omega_s \left(t - \frac{F}{2}\right)\right] \sin s\pi x \quad (86)$$

If a wave having a front so slow that it causes no perceptible oscillations is chopped on the tail, the subsequent distribution will consist of the oscillations corresponding to an infinite rectangular wave $(-E)$, but these oscillations will take place about the zero axis instead of the $(-xE)$ axis.

Typical Lightning Wave, Fig. 112F.—In the case of a wave with exponential front and tail

$$E(t) = E\,(\epsilon^{-at} - \epsilon^{-bt}) \quad (87)$$

and (71) then yields

$$e = E\,(\epsilon^{-at} - \epsilon^{-bt})\,x + E\sum_{1}^{\infty} A_s \sin s\pi x \left[\frac{a^2\,\epsilon^{-at}}{a^2 + \omega_s^2} - \frac{b^2\,\epsilon^{-bt}}{b^2 + \omega_s^2} + \frac{\omega_s}{\sqrt{a^2 + \omega_s^2}} \cos\left(\omega_s t + \tan^{-1}\frac{a}{\omega_s}\right) - \frac{\omega_s}{\sqrt{b^2 + \omega_s^2}} \cos\left(\omega_s t + \tan^{-1}\frac{b}{\omega_s}\right)\right] \quad (88)$$

A wave with an infinite tail and exponential front is given by putting $a = 0$, and (88) reduces to

$$e_{a=0} = E\,(1 - \epsilon^{-bt})\,x + E\sum_{1}^{\infty} A_s \sin s\,\pi\,x \left[-\frac{b^2\,\epsilon^{-bt}}{b^2 + \omega_s^2} + \frac{b}{\sqrt{b^2 + \omega_s^2}} \cos\left(\omega_s t - \tan^{-1}\frac{\omega_s}{b}\right)\right] \quad (89)$$

The reduction factor

$$\frac{b}{\sqrt{b^2 + \omega_s^2}} = \frac{1}{\sqrt{1 + (\omega_s\,b^2)}} \cong \frac{1}{\sqrt{1 + \left(\frac{2\,\pi\,F}{3\,T_s}\right)^2}} \quad (90)$$

has been plotted in Fig. 113C. The steeper the wave front the higher the amplitudes of oscillation. The higher harmonics are wiped out first by a depression of wave front. If $b = \infty$ the wave front is perpendicular and the solution (89) then reverts to that for an infinite rectangular wave.

A wave with a perpendicular front and an exponential tail is given by putting $b = \infty$, and (88) then reduces to

$$e = xE\epsilon^{-at} + E\sum_{1}^{\infty} A_s \sin s\pi x \left[\frac{a^2\epsilon^{-at}}{a^2+\omega_s^2} + \frac{\omega_s}{\sqrt{a^2+\omega_s^2}}\cos\left(\omega_s t + \tan^{-1}\frac{a}{\omega_s}\right)\right] \quad (91)$$

The reduction factor, Fig. 114, is

$$\frac{\omega_s}{\sqrt{a^2+\omega_s^2}} = \frac{1}{\sqrt{1+(a/\omega_s)^2}} \cong \frac{1}{\sqrt{1+(0.11\,T_s/L)^2}} \quad (92)$$

Thus in the case of a falling tail it is the lower harmonics which are wiped out first, rather than the higher harmonics as in the case of an increased wave front. However, as seen from Fig. 114, the wave length must be very short to affect the fundamental seriously.

If the tail is infinite, $a = 0$, the solution (91) reverts to that for an infinite rectangular wave.

For practical estimates of the effect of a wave of given front and tail, the two effects may be calculated separately and multiplied together. For example, consider a grounded neutral transformer characterized by $\alpha = 10$, $f_1 = 10{,}000$, $\gamma_1 = 0.003$, and $\sigma = 20$ subjected to the impact of a traveling wave with a 5-ms. exponential front and a 20-ms. tail. Then the following table illustrates the use

Fig.	Harmonic (s)	1	2	3	4	5	6
108	Decrement (γ_s)	0.003	0.006	0.011	0.018	0.027	0.038
107	Natural period ($2\pi/\omega_s$)	100	28.0	14.6	9.6	7.1	5.7
06	Amplitude (A_s)	−0.58	+0.23	−0.12	+0.07	−0.03	+0.02
13C	Front factor	0.99	0.94	0.82	0.68	0.56	0.49
14	Tail factor	0.88	0.99	1.00	1.00	1.00	1.00
	Damping $\epsilon^{-\gamma_s T_1/2}$	0.86	0.74	0.58	0.41	0.26	0.15
	Reduced amplitudes	−0.44	0.16	−0.06	0.02	0.00	0.00

of the several sets of curves which have been derived. The reduced amplitudes are calculated as the product of A_s and the reduction factors for front, tail, and losses, the last effect to extend over a half period of the fundamental.

In addition to these reduced amplitudes of oscillation, the axis has been steadily declining, so that the transient distribution does not exceed the linear distribution $x\,E$, except in the immediate vicinity of the neutral.

However, for waves which are long compared with the natural period of the fundamental, abnormal voltages of the order of 1.4 E

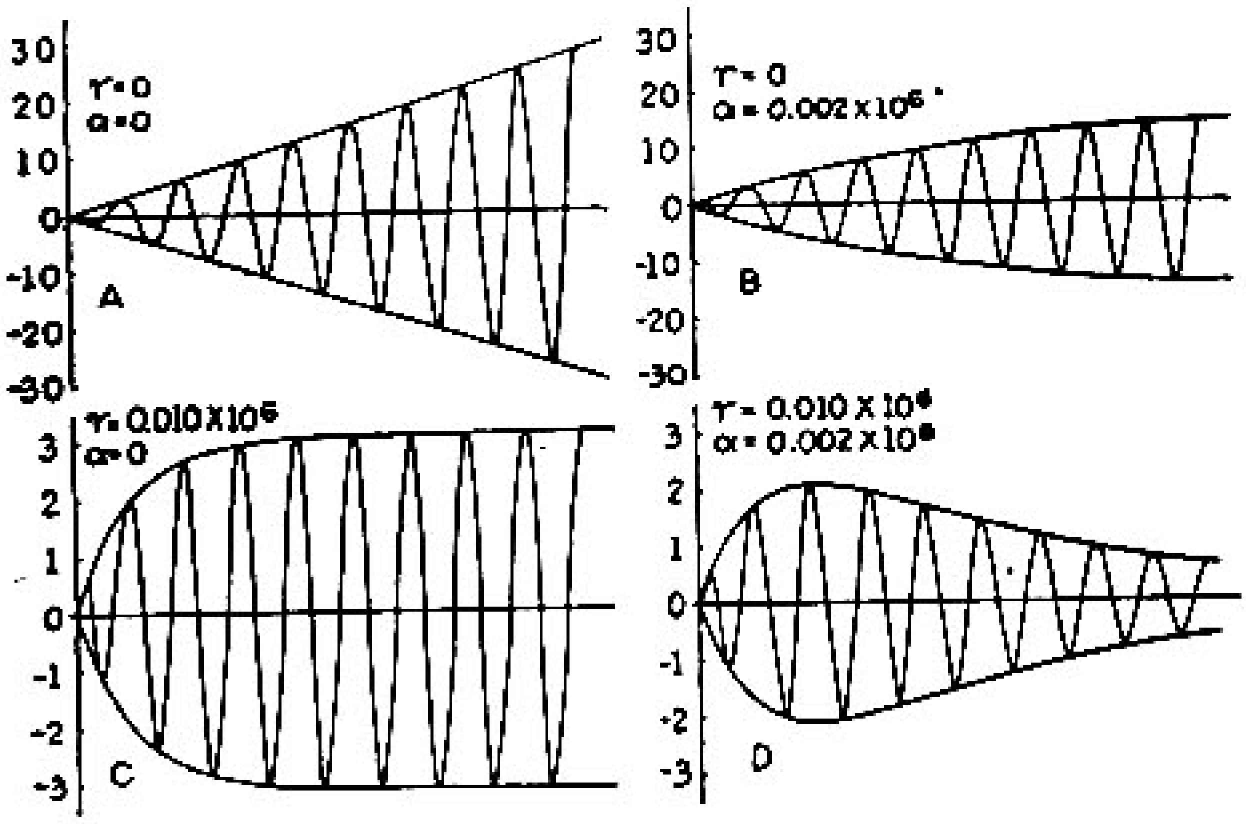


FIG. 117.—Effect of Decrement Factors on Cumulative Oscillations
γ = transformer decrement, a = applied wave decrement

may be built up in the neighborhood of a quarter of the way from the line end.

Damped Oscillatory Waves, Fig. 112*G*.—When a sustained oscillatory wave (alternating current) is in resonance with one of the natural frequencies of a transformer, the amplitudes of that harmonic build up indefinitely at a linear rate, as shown in Fig. 117*A*. If, however, there is a finite decrement in the applied wave, then the amplitude of the resonant frequency reaches a distinct upper limit, at which it persists forever if there are no internal losses in the transformer to damp it out. This is illustrated in Fig. 117*B*, where the presence of even a moderate decrement in the applied wave has resulted in limiting the voltage rise to 14 times the first crest of Fig. 117*A*. When the applied wave is sustained, but the natural oscillations of the transformer are damped by the losses of the transformer, then again the cumulative oscillations are definitely limited as shown in Fig. 117*C*. If, now, there are decrements in both the applied wave and in the natural-frequency oscillation, then the cumulative oscillations reach a maximum beyond which they decrease ultimately to zero, as shown in Fig. 117*D*. This characteristic has been verified by cathode-ray oscillograms of resonant oscillations in transformers, of which Fig. 118 is an example.

In order to estimate quickly the effect of the decrement factors in the applied wave and natural oscillation in limiting the maximum of cumulative oscillations, the curves of Fig. 119 have been prepared. These curves give the maximum crest of the envelope of oscillations corresponding to the damped oscillatory wave $E\,\epsilon^{-at}\sin(bt+\theta)$ when the natural frequency oscillation of the transformer is $A\,\epsilon^{-\gamma t}\cos bt$.

Substituting in (70)

$$E(t) = E\,\epsilon^{-at}\sin(bt+\theta)$$

there results

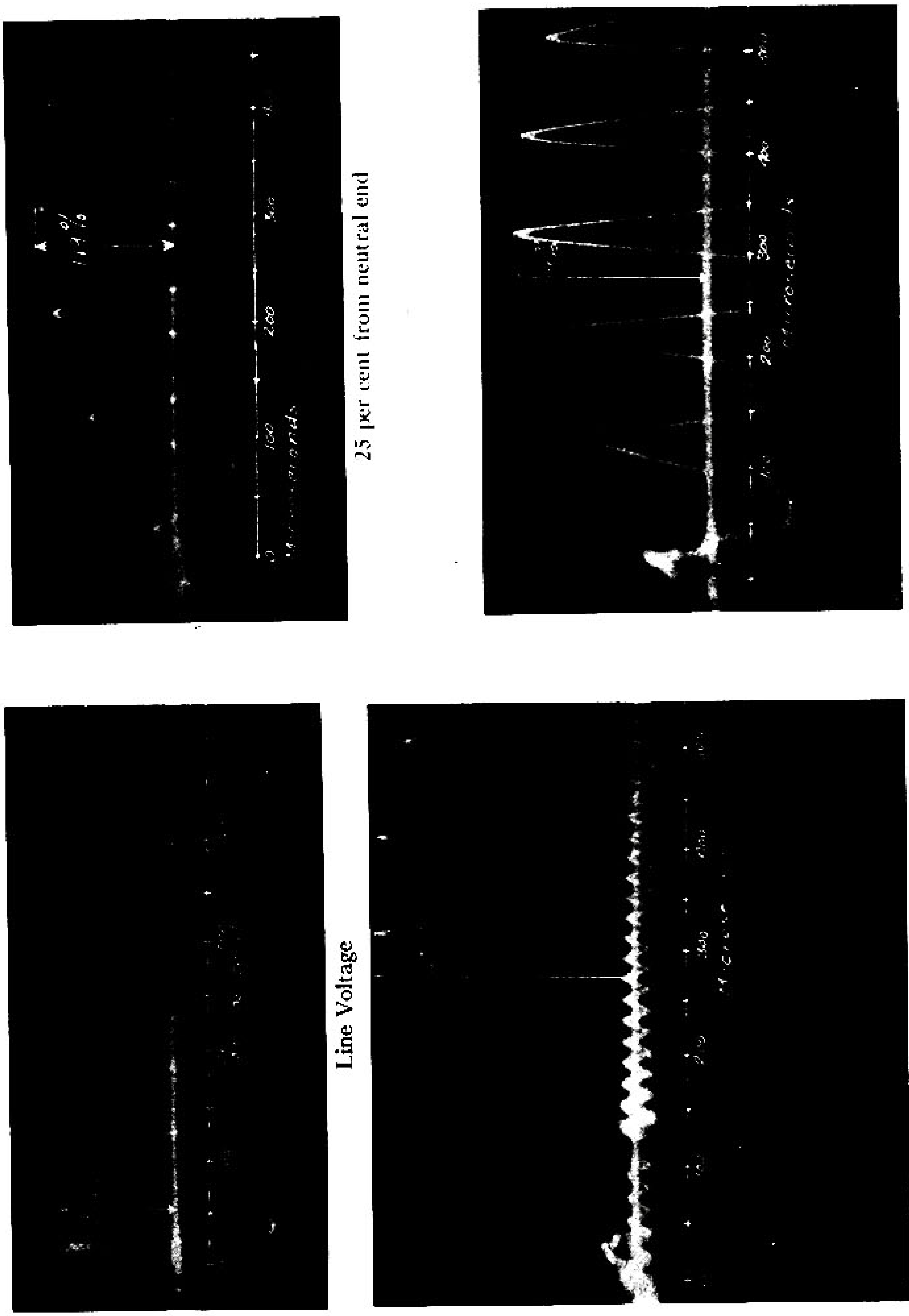
$$\begin{aligned} e = xE\,\epsilon^{-at}\sin(bt+\theta) + \sum_1^\infty A_s \sin s\pi x \Bigg\{ & E\,\epsilon^{-at}\sin(bt+\theta) \\ & + \frac{E}{2}\sqrt{\frac{\omega_s^2+\gamma_s^2}{(a-\gamma_s)^2+(b-\omega_s)^2}}\,[\epsilon^{-at}\sin(bt+\theta-\lambda_s) - \epsilon^{-\gamma_s t} \\ & \qquad \sin(\omega_s t+\theta-\lambda_s)] \\ & + \frac{E}{2}\sqrt{\frac{\omega_s^2+\gamma_s^2}{(a-\gamma_s)^2+(b+\omega_s)^2}}\,[\epsilon^{-at}\sin(bt+\theta+\psi_s) + \epsilon^{-\gamma_s t} \\ & \qquad \sin(\omega_s t-\theta-\psi_s)]\Bigg\} \end{aligned} \quad (93)$$

where

$$\lambda = \tan^{-1}\left(\frac{\omega}{\gamma}\right) + \tan^{-1}\left(\frac{\omega-b}{a-\gamma}\right) = \tan^{-1}\left[\frac{a\omega-b\gamma}{\gamma(a-\gamma)-\omega(\omega-b)}\right] \quad (94)$$

$$\psi = \tan^{-1}\left(\frac{\omega}{\gamma}\right) + \tan^{-1}\left(\frac{\omega+b}{a-\gamma}\right) = \tan^{-1}\left[\frac{a\omega+b\gamma}{\gamma(a-\gamma)-\omega(\omega+b)}\right] \quad (95)$$

The condition of principal interest is when the applied wave has a frequency equal, or very nearly equal, to that of one of the natural frequencies of oscillation of the transformer, for under that condition there is a possibility of building up excessive internal voltages by cumulative or forced oscillations. Suppose that for a particular har-



Line Voltage

25 per cent from neutral end

50 per cent from neutral end

75 per cent from neutral end

FIG. 118.—Cumulative Oscillations in Transformers

monic $\omega = b$. Then for that harmonic the terms in braces in Equation (93) become (dropping the subscript s)

$$\Bigg\{E\,\epsilon^{-at}\sin(bt+\theta) + \frac{E}{2}\sqrt{\frac{b^2+\gamma^2}{(a-\gamma)^2}}\,(\epsilon^{-at}-\epsilon^{-\gamma t})\sin(bt+\theta-\lambda)$$

$$+\frac{E}{2}\sqrt{\frac{b^2+\gamma^2}{(a-\gamma)^2+4\,b^2}}\left[\epsilon^{-at}\sin(bt+\theta+\psi)+\epsilon^{-\gamma t}\sin(bt-\theta-\psi)\right]\Bigg\} \tag{96}$$

and

$$\lambda = \tan^{-1}\left(\frac{b}{\gamma}\right) \tag{97}$$

$$\psi = \tan^{-1}\left(\frac{b\,(a+\gamma)}{\gamma\,(a-\gamma)-2\,b^2}\right) \tag{98}$$

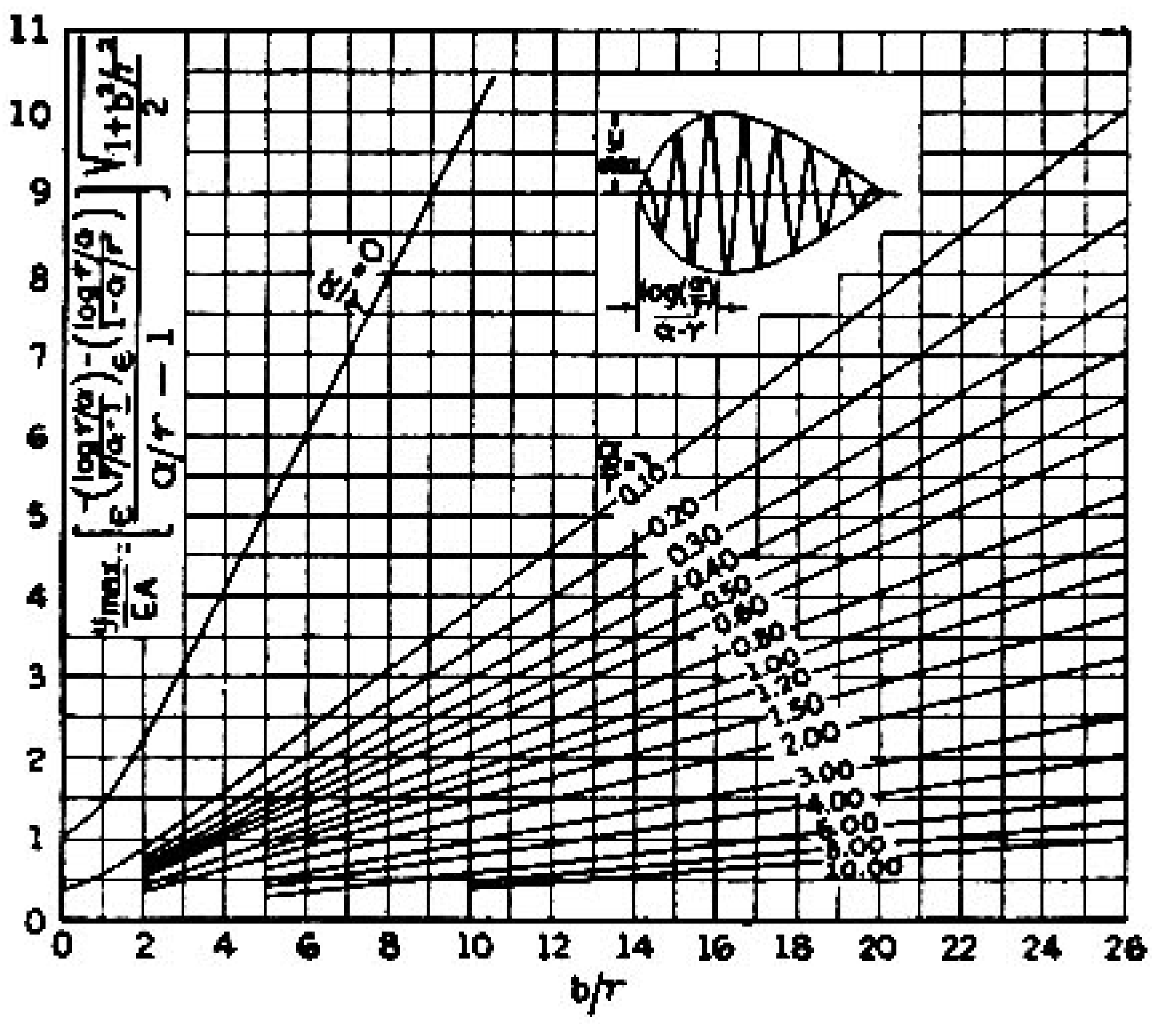


FIG. 119.—Maximum of the Envelope of Cumulative Oscillations

Applied Wave: $E\epsilon^{-at}\sin(bt+\theta)$
Natural Oscillation: $A\epsilon^{-\gamma t}\cos bt$

The second term of Equation (96) is responsible for the excessive rise in voltage due to cumulative oscillations. The term consists of a sinusoidal oscillation within the envelope of

$$y = \frac{E}{2}\sqrt{b^2+\gamma^2}\left(\frac{\epsilon^{-at}-\epsilon^{-\gamma t}}{a-\gamma}\right) = \frac{E}{2}\sqrt{\frac{b^2}{\gamma^2}+1}\;\frac{\epsilon^{-at}-\epsilon^{-\gamma t}}{(a/\gamma)-1} \tag{99}$$

Differentiating Equation (99) and equating to zero as the condition for a maximum, there is

$$\frac{dy}{dt} = 0 = -a\,\epsilon^{-at} + \gamma\,\epsilon^{-\gamma t} \tag{100}$$

Therefore

$$t = \frac{\log\,(\gamma/a)}{\gamma - a} = t' \tag{101}$$

Therefore

$$y_{\max} = \frac{E}{2}\sqrt{\frac{b^2}{\gamma^2} + 1}\;\frac{\epsilon^{-at'} - \epsilon^{-\gamma t'}}{(a/\gamma) - 1} \tag{102}$$

This expression is useful for estimating the maximum voltages that can be expected, although exact values must be calculated from the complete Equation (96). It can be expressed as a function of the ratio (γ/a) as follows:

$$at' = a\,\frac{\log\,(\gamma/a)}{\gamma - a} = \frac{\log\,(\gamma/a)}{(\gamma/a) - 1} \tag{103}$$

$$\gamma t' = \gamma\,\frac{\log\,(\gamma/a)}{\gamma - a} = \frac{\log\,(\gamma/a)}{1 - (a/\gamma)} \tag{104}$$

Curves for $(y_{\max}/E)$ have been plotted against the ratio (b/γ) in Fig. 119 with the ratio (a/γ) as parameter. Ordinarily, (b/γ) is so large compared with unity that

$$y_{\max} \cong \frac{E}{2}\,\frac{b}{\gamma}\,\frac{(\epsilon^{-at'} - \epsilon^{-\gamma t'})}{(a/\gamma) - 1} \tag{105}$$

Thus the envelope of cumulative oscillations has an amplitude directly proportional to the resonant frequency, but since the decrement also increases with the frequency, it does not follow that the amplitude will necessarily increase with the frequency.

There are a number of special cases of sufficient interest to warrant detailed consideration.

Case I.—If the applied wave is sustained $(a = 0)$ and the transformer is free of losses $(\gamma = 0)$, then by Equations (97) and (98)

$$\lambda = \tan^{-1}(\infty) = 90 \text{ deg.} \tag{106a}$$

$$\psi = \tan^{-1}(-0) = 180 \text{ deg.} \tag{106b}$$

also

$$\left.\frac{\epsilon^{-at} - \epsilon^{-\gamma t}}{a - \gamma}\right/_{a \to \gamma \to 0} = -t\,\epsilon^{-\gamma t} = -t \tag{107}$$

Substituting Equations (106*a*), (106*b*), and (107) into (96), there is

$$\left\{E \sin (bt + \theta) + \frac{E}{2} bt \cos (bt + \theta) - \frac{E}{4}[\sin (bt + \theta) + \sin (bt - \theta)]\right\}$$

$$= E\left\{\frac{3}{4} \sin (bt + \theta) - \frac{1}{4} \sin (bt - \theta) + \frac{bt}{2} \cos (bt + \theta)\right\} \quad (108)$$

$$= \frac{E}{2} (\sin bt + bt \cos bt) \text{ for } \theta = 0 \quad (108a)$$

$$= \frac{E}{2} (2 \cos bt - bt \sin bt) \text{ for } \theta = 90^\circ \quad (108b)$$

Thus the amplitude of the oscillation increases linearly without limit until the breakdown of the insulation. The phase angle θ has considerable effect on the initial part of the oscillations, but does not influence the ultimate limit or the rate at which it is approached. Equation (108*a*) has been plotted in Fig. 117*A*. Actually, of course, this case is only of theoretical interest, because all transformers have losses and definite decrement factors.

Case II.—If a transformer having zero losses ($\gamma = 0$) is subjected to a damped incident wave, then Equation (96) reduces to

$$E\left\{\epsilon^{-at} \sin (bt + \theta) - \frac{b}{2}\frac{\epsilon^{-at} - 1}{a} \cos (bt + \theta)\right.$$
$$\left. + \frac{b}{2}\sqrt{\frac{1}{a^2 + 4b^2}}[\epsilon^{-at} \sin (bt + \theta + \psi) + \sin (bt - \theta - \psi)]\right\} \quad (109)$$

Here the envelope of oscillation approaches a maximum at $t = \infty$ of

$$y_{\max} = \frac{E\,b}{2\,a} \quad (110)$$

When $\theta = 0$ and $(b\ a)$ is so large (as is usually the case) that $\psi \cong 180°$, then Equation (109) simplifies to

$$E\left\{\frac{3\,\epsilon^{-at} - 1}{4} \sin bt - \frac{b}{2}\left(\frac{\epsilon^{-at} - 1}{a}\right) \cos bt\right\} \quad (111)$$

This equation has been plotted in Fig. 117*B* for $b = 0.02\,\pi \times 10^6$ and $a = 0.002 \times 10^6$. The oscillations build up (theoretically at $t = \infty$) to a sustained maximum of

$$\frac{E}{2}\left\{\frac{b}{a} \cos bt - \frac{1}{2} \sin bt\right\} \quad (112)$$

In other words, the presence of a decrement in the applied wave has definitely limited the rise due to cumulative oscillations. But once this maximum is reached it is *sustained*, that is, continues even after the applied wave has decayed to zero.

Case III.—When a transformer having losses (γ finite) is subjected to a sustained oscillatory wave ($a = 0$) the resulting form of Equation (96) is

$$E\left\{\sin(bt + \theta) + \sqrt{\frac{b^2 + \gamma^2}{4\gamma^2}}\,(1 - \epsilon^{-\gamma t})\sin(bt + \theta - \lambda)\right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{b^2 + \gamma^2}{4\,b^2 + \gamma^2}}\,[\sin(bt + \theta + \psi) + \epsilon^{-\gamma t}\sin(bt - \theta - \psi)]\right\} \quad (113)$$

But γ is small compared to b, so that $\lambda \cong 90°$ and $\psi \cong 180°$. Thus Equation (113) simplifies to

$$E\left\{\frac{3 - \epsilon^{-\gamma t}}{4}\sin bt - \frac{b}{2\,\gamma}(1 - \epsilon^{-\gamma t})\cos bt\right\} \quad (114)$$

As in the previous case, the oscillations build up to a definite sustained limit, reaching an ultimate maximum of

$$\frac{E}{2}\left\{\frac{3}{2}\sin bt - \frac{b}{\gamma}\cos bt\right\} \quad (115)$$

Equation (115) has been plotted in Fig. 117*C* for $\gamma = 0.010 \times 10^6$ and $b = 0.02\,\pi \times 10^6$.

Case IV.—If $a = \gamma$, then by Equation (99) the envelope factor becomes an indeterminate which is evaluated as

$$y = E\sqrt{b^2 + \gamma^2}\,\frac{\epsilon^{-at} - \epsilon^{-\gamma t}}{a - \gamma}\Big/_{a=\gamma} = -\frac{E}{2}\sqrt{b^2 + \gamma^2}\,t\,\epsilon^{-\gamma t} \quad (116)$$

Hereby Equation (96) becomes

$$\left\{E\sin(bt + \theta) - \frac{E}{2}\sqrt{b^2 + \gamma^2}\,t\sin(bt + \theta - \lambda)\right.$$
$$\left. + \frac{E}{2\,b}\sqrt{b^2 + \gamma^2}\sin bt\cos(\theta + \psi)\right\}\epsilon^{-\gamma t} \quad (117)$$

Making the usual substitutions for γ small compared to b and $\theta \cong 0$, this equation is reduced to

$$E\,\frac{\epsilon^{-\gamma t}}{2}(\sin bt + bt\cos bt) \quad (118)$$

The envelope reaches a maximum at $t = 1/\gamma$ of

$$y_{\max} = \frac{Eb}{2\gamma\epsilon} \tag{119}$$

This is less than that occurring in Case III by the ratio $1/\epsilon = 0.368$; and moreover the maximum is not sustained, but finally decays to zero, as is true for all cases in which both the natural oscillations and the applied waves have finite decrements.

Case V.—When there are finite decrements in both the applied wave and in the natural oscillation of the transformer—the general and practical case—then Equation (96) applies. However, since γ/b is a small quantity, the equation simplifies to

$$E\left\{\frac{3}{4}\,\epsilon^{-at}\sin(bt+\theta) - \frac{1}{4}\,\epsilon^{-\gamma t}\sin(bt-\theta) - \frac{b}{2}\left(\frac{\epsilon^{-at}-\epsilon^{-\gamma t}}{a-\gamma}\right)\cos(bt+\theta)\right\} \tag{120}$$

and if $\theta \cong 0$ it may be further condensed to

$$E\left\{\left(\frac{3\,\epsilon^{-at}-\epsilon^{-\gamma t}}{4}\right)\sin bt - \frac{b}{2}\left(\frac{\epsilon^{-at}-\epsilon^{-\gamma t}}{a-\gamma}\right)\cos bt\right\} \tag{121}$$

This equation has been plotted in Fig. 117*D* for $a = 0.002 \times 10^6$, $\gamma = 0.010 \times 10^6$, and $b = 2\pi \times 10^4$. Fig. 118 shows oscillograms of the oscillations in an actual transformer, for which the applied wave is in resonance with the fundamental natural period. It will be noticed that the equations correctly depict the phenomenon as an oscillation confined by an envelope specified by the difference of two exponentials. Moreover, the numerical agreement between this oscillogram and calculations is excellent in all details.

The dominating effect of the decrements in limiting the maximum internal voltages obtainable by damped oscillatory incident waves is clearly demonstrated by the four curves of Fig. 117. When there are decrements neither in the applied wave nor in the natural oscillation of the transformer, the voltage rapidly builds up to destructive values. For instance, after 10 cycles the voltage has built up linearly to 31 times the amplitude of the applied wave. However, the presence of a relatively small decrement in the applied wave, Fig. 117*B*, limits the rise to half that value. With a sustained oscillation applied to the transformer terminal, the normal decrement of the transformer oscillation may be sufficient to hold the rise to three times, whereas with both decrements present the rise may not exceed two times.

The envelopes of the voltage distributions in a transformer, corresponding to an infinite rectangular wave and a damped oscillatory

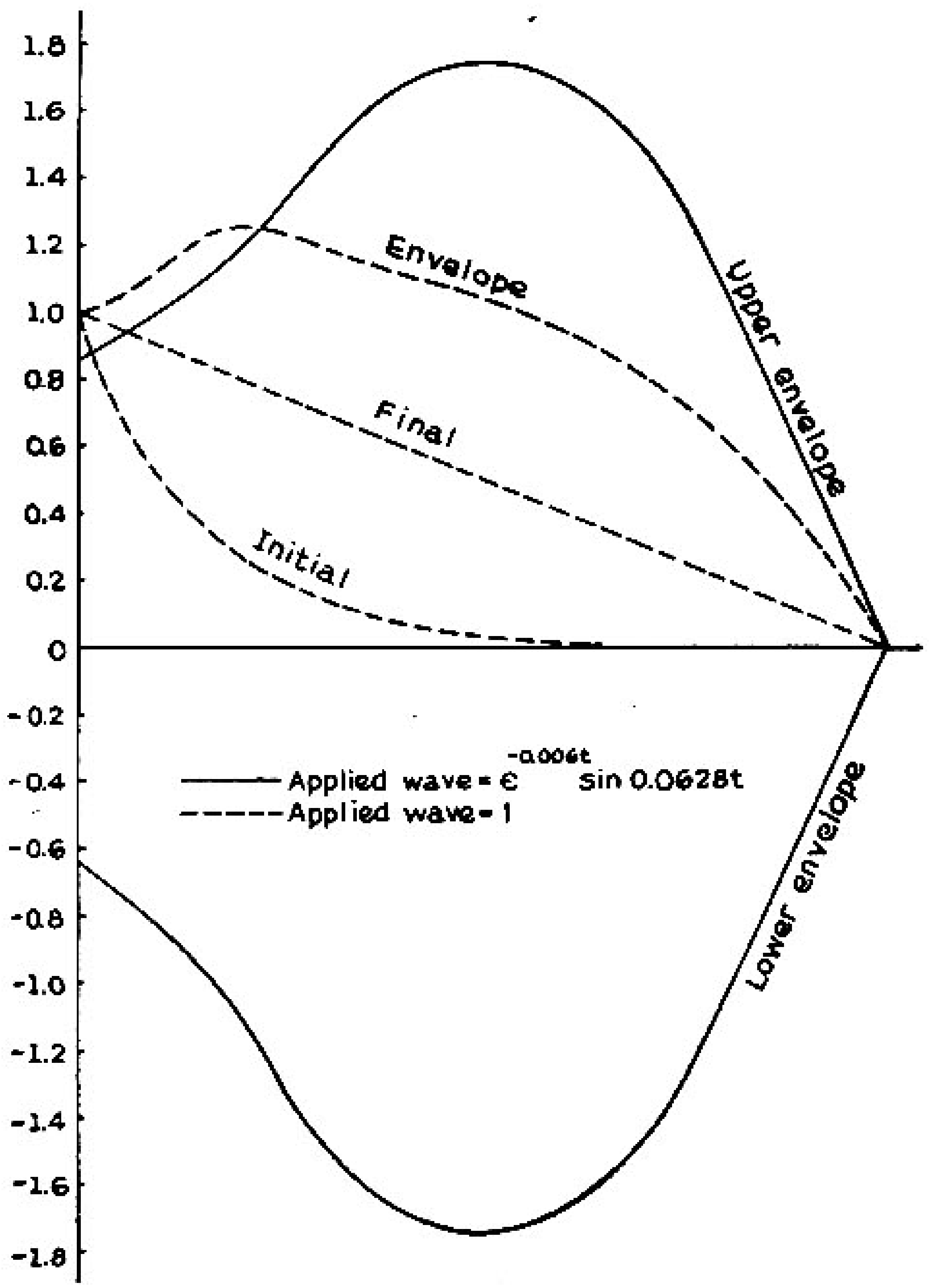


FIG. 120.—Transformer Oscillations Due to Damped Sinusoidal and Infinite Rectangular Waves

wave respectively, are shown in Fig. 120. These particular distributions were based on the following data and assumptions:

Applied damped oscillatory wave $E\,\epsilon^{-at}\sin(bt+\theta) = \epsilon^{-0.006t}$ $\sin(0.02\,\pi\,t)$ (in resonance with the fundamental natural frequency).

Initial distribution for $\alpha = 10$, Fig. 105.

Amplitudes of oscillation for $\alpha = 10$, Fig. 106.

Fundamental natural frequency of oscillation $f_1 = 10{,}000$.

Harmonic frequencies f_s from Fig. 107.

Decrement of the fundamental frequency $\gamma_1 = 0.002 \times 10^6$.
Decrement of the harmonic frequencies γ_s from Fig. 108 for $\sigma = 20$.
The comparison is facilitated by the following tabulation:

Applied wave	E	$E\epsilon^{-at} \sin (bt + \theta)$
Initial distribution	$E \dfrac{\sinh \alpha x}{\sinh \alpha}$	$E \sin \theta \left(\dfrac{\sinh \alpha x}{\sinh \alpha} \right) = 0$
Final distribution	xE	0
Maximum voltage	$+1.24\,E$ at $t = 40$	$+1.75\,E$ at $t = 300$ $-1.75\,E$ at $t = 250$

Because of the high decrement factors of the harmonics and the long time required for the maximum to occur in the case of the applied damped oscillatory wave, only the fundamental with which it was in resonance contributed to the maximum voltage to ground. The second harmonic amounted to only 3 per cent by that time, but it added less than 1 per cent to the maximum (because it was not in space phase), and therefore was negligible. The maximum terminal voltage is 0.862 E, so that the ratio of the crest of the envelope of oscillation to the crest of the highest loop of the applied wave is 203 per cent. Thus for equal maximum terminal voltages the damped oscillatory wave causes 64 per cent higher internal stresses than the infinite rectangular wave. Moreover, since the oscillation reverses to full negative value, the gradients along the winding are much more severe and the insulation is subjected to over three times the *range* in stresses. Considering typical lightning waves with falling tails, instead of infinite rectangular waves, it may be stated that for equal terminal voltages, a damped sinusoidal wave in resonance with the fundamental natural frequency of the transformer will cause internal voltages approximately twice as great as the lightning wave. Fortunately, however, the oscillatory traveling waves which have so far been recorded in the field (due to switching) do not exceed 5.5 times the normal line-to-neutral voltage of the transmission system.

Waves Applied Simultaneously at Line and Neutral Terminals.—The previous analysis supposed the applied wave at the line end only, and the neutral either solidly grounded or completely isolated. But in practice the transformer may be one of a delta-connected three-phase bank and therefore subject to incident waves at either or both terminals; or the neutral may be grounded through an impedance.

A rigorous solution for arbitrary impedances in the neutral has not been obtained, but it is shown in Chapter XV that the neutral transient can be calculated with precision without solving the complicated distributed network of the transformer. Once the neutral voltage has been determined, according to the method of Chapter XV, it may be regarded as an applied wave at the neutral terminal and the complete solution for the internal oscillations found by the principle of superposition; for if $E(t)$ and $E_n(t)$ are the line and neutral voltages respectively, and

$$e_1\,(x,\,t) = \text{internal transient voltage corresponding to } E(t) \text{ with neutral end grounded} \tag{122}$$

$$e_2\,(1-x,\,t) = \text{internal transient voltage corresponding to } E_n(t) \text{ with line end grounded} \tag{123}$$

then the complete solution is

$$e(x,t) = e_1(x,t) + e_2(1-x,t) \tag{124}$$

where the $(1 - x)$ implies that distance along the stack is measured from the line end when the applied voltage is at the neutral.

As an example, let

$$\left.\begin{aligned} E(t) &= E \text{ (infinite rectangular wave)} \\ E_n(t) &= E\,(1 - \epsilon^{-bt}) \end{aligned}\right\} \tag{125}$$

This neutral voltage corresponds to a resistance in the neutral. By (44a) and (89), respectively

$$e_1(x,t) = xE + E\sum_1^{\infty} A_s \cos \omega_s t \cdot \sin s\,\pi\,x \tag{126}$$

$$e_2(1-x,t) = (1 - x)E(1 - \epsilon^{-bt}) + E\sum_1^{\infty} A_s \sin s\,\pi\,(1 - x)$$
$$\left[-\frac{b^2\,\epsilon^{-bt}}{b^2 + \omega_s^2} + \frac{b}{\sqrt{b^2 + \omega_s^2}} \cos\left(\omega_s t - \tan^{-1}\frac{\omega_s}{b}\right)\right] \tag{127}$$

and the complete solution is the sum of (126) and (127).

Fig. 121 illustrates the effect of similar exponential waves $E\,\epsilon^{-\alpha t}$ striking both ends of a winding simultaneously. The calculation applies to a transformer having a fundamental natural period of 10,000 cycles (grounded neutral) and $\alpha = 10$, subjected to the impact of 40-ms. exponential waves. It is evident from Fig. 114 that the principal harmonic amplitudes of oscillation are not much reduced by the 40-ms. wave, but the decline of the axis of oscillation has considerable effect on the crest of the envelope of oscillations. Thus by the time the oscillatory components are making their greatest contributions (approximately at $t = 40$ ms.) the axis of oscillations has declined to half value and the crest of the envelope of oscillations is therefore $0.5\,E$ less than it would have been with infinite rectangular applied waves. It will be seen that the even harmonics cancel and

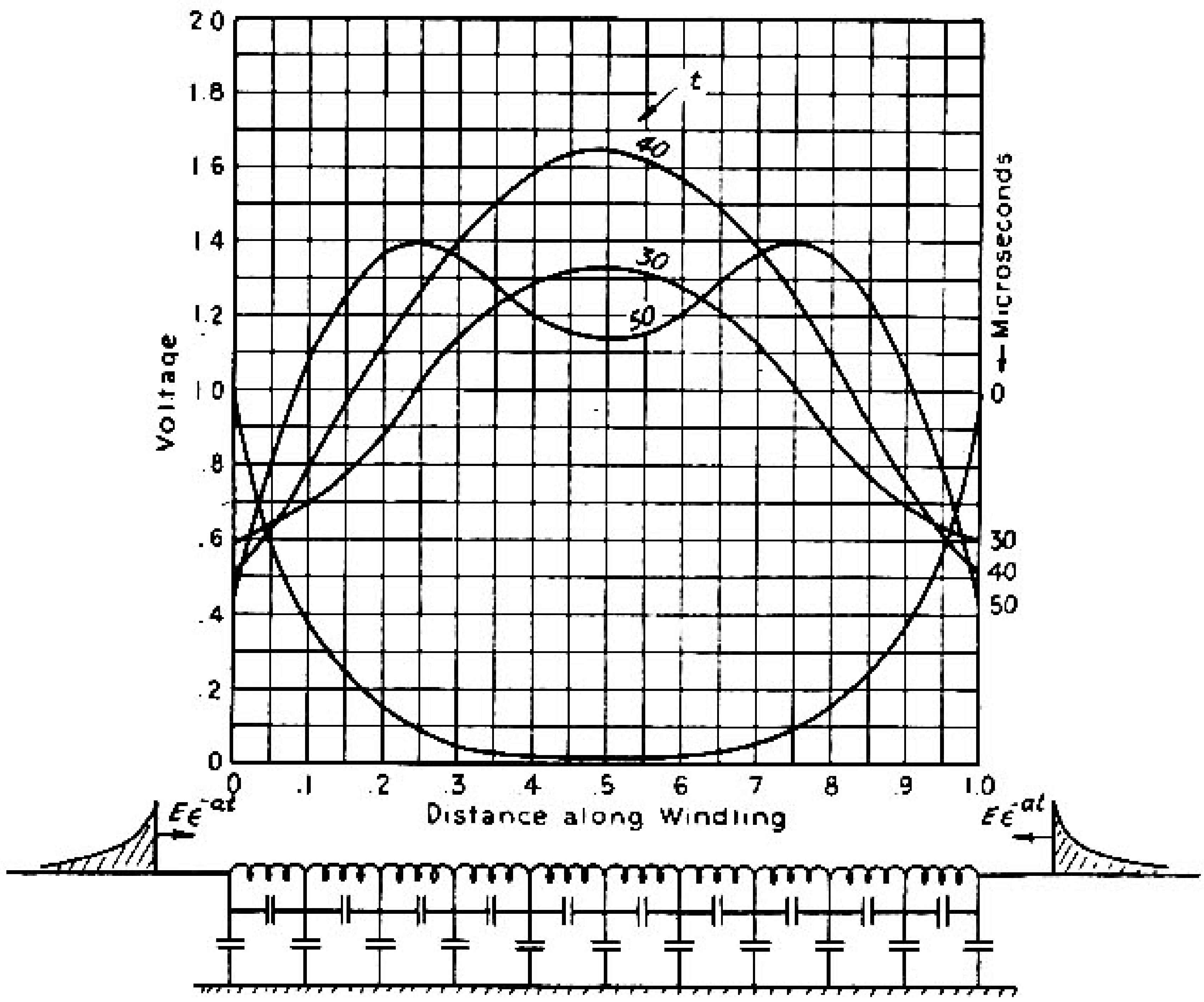


FIG. 121.—Exponential Waves Applied Simultaneously to Both Terminals

the odd harmonics double, for equal waves simultaneously applied at both terminals.

SUMMARY OF CHAPTER XIV

The solutions given in Chapters XII and XIII are for infinite rectangular waves. The corresponding solutions for waves of arbitrary shape are given by Duhamel's theorem, which yields the general expressions of Equation (70). As special cases thereof are the wave shapes of Fig. 112. The amplitudes of oscillation caused by these waves are obtained from those corresponding to an infinite rectangular wave upon multiplying by the proper reduction factor, as follows:

Wave Shape	Reduction Factor	Curves
Triangular front	Equation (84)	Fig. 113A
Sinusoidal front	....	Fig. 113B
Exponential front	Equation (90)	Fig. 113C
Finite rectangular	Equation (75	Fig. 114
Triangular tail	....	Fig. 114
Exponential tail	Equation (92)	Fig. 114
Exponential front and tail	Equation (88)	
Damped oscillatory wave	Equation (102)	Fig. 119

From the equations and curves it is evident that:

1. Lengthening the wave front decreases the amplitudes of oscillation, the reduction being greater for the higher harmonics, and if the wave front is long enough to eliminate the fundamental all the higher harmonics become negligible.
2. Shortening the wave tail other than by "chopping" decreases the amplitudes, the reduction being greater for the lower harmonics, so that the fundamental is the first to be eliminated.
3. Chopping a wave on the tail, as by abrupt insulator flashover or lightning-arrester operation, may increase the amplitudes of a number of harmonics, with the possibility of doubling the amplitude of any harmonics for which the wave length is an odd multiple of its natural period.
4. Damped oscillatory waves in resonance with a natural period of oscillation develop cumulative oscillations, limited only by the losses in the transformer and the decrement in the applied wave. In practical cases the amplitudes are about twice as high as those caused by infinite rectangular waves of equal crest voltage.
5. From the curves for reduction factors and the curves given in the previous chapter for amplitudes, frequencies, and decrements of oscillations, the transient oscillations may be quickly computed for waves of arbitrary shape.

When waves are applied simultaneously at both terminals of a transformer winding, the internal transient may be found by superposition, that is, as the instantaneous sum of the transients caused by each applied wave separately with the other terminal grounded.

CHAPTER XV

TERMINAL TRANSIENTS

Synthesis of Equivalent Circuits.—The general equations of Chapter XII apply only to windings whose terminals are either directly grounded or open-circuited, and consequently it is possible to calculate from them only the two extreme limits of the terminal transients. Moreover, even the calculation for a grounded or isolated terminal is a very laborious job. It becomes necessary, therefore, to improvise approximate equivalent circuits for the ready calculation of terminal transients, because these terminal transients are impressed upon the connected apparatus and may be a source of danger to that equipment.

The building up of an appropriate equivalent circuit, in the absence of a rigorous and comprehensive solution, is a matter of some intuition. There are always certain limiting conditions which must be satisfied, and experience usually suggests which factors are of major and which are only of minor or subsidiary importance. Finally, it is essential to fall back on test results and experimental evidence for verification. In the present case the general equations of Chapter XII will furnish the limiting condition that must be satisfied. A very general and complex equivalent circuit will first be established, and then, from a critical study of the importance and rôle of the different elements of that circuit, very much simplified equivalent circuits applicable to specific conditions will be obtained. It goes without saying that a *practical* equivalent circuit must be simple enough so that it can be solved analytically under the conditions to which it applies.

From the general distributed circuit of Fig. 101 it is evident that there are a number of distinct paths connecting the terminals, and to ground. Thus by inspection the following paths are evident:

a. From primary line terminal to primary neutral through the inductance of the winding. If the secondary winding is carrying current this path will be influenced by the mutual inductance between windings.

b. Between the secondary terminals through the inductance of the secondary winding, and influenced by the mutual inductance between windings.

c. Between primary terminals and between secondary terminals through the series capacitances. The total series capacitances K_1 and K_2 are not entirely replaceable by lumped capacitances, except in the case of a non-resonating transformer.

d. Effective capacitances from terminals to ground.

e. Effective capacitance between primary and secondary windings.

f. Composite paths, partly through the inductances and partly through the capacitances, giving rise to the natural oscillations. An isolated neutral oscillates at a definite fundamental natural period which may be calculated from the general equations or determined experimentally.

Fig. 122 shows the simplest possible circuit having the necessary degrees of freedom to conform to the above terminal effects. If the neutrals are isolated ($Z_1 = Z_2 = \infty$), the fictitious capacitances C_1' and C_2' in the equivalent circuit cause the neutral voltages to oscillate at their fundamental natural frequencies.

At the instant of impact of the traveling wave the voltages depend only on the capacitance network and the terminal impedances. If these impedances are infinite (open circuits), the circuit effective at the first instant is that shown in Fig. 122*b* with the switches s_1, s_2, and s_3 open; and for grounded terminals the switches are closed. Now the effective capacitance of a transformer with respect to the line terminal is

$$\left.\begin{aligned} C_0 &\cong \sqrt{CK} && \textit{for an ordinary transformer} \\ &= \left(\frac{C}{2} + K\right) && \textit{for a non-resonating transformer} \end{aligned}\right\}$$

which must be equal to the effective capacitance of the equivalent circuit, Fig. 122*b*, with the switches closed, that is

$$\left.\begin{aligned} C_{01} &= C_1'' + K_1' + C_3' && \textit{for the primary} \\ C_{02} &= C_2'' + K_2' + C_3' && \textit{for the secondary} \end{aligned}\right\} \qquad (1)$$

If $Z_1 = Z_2 = \infty$ and an abrupt voltage is applied to a line terminal, the initial voltage which appears at the corresponding neutral can be calculated by the methods of Chapter XII; and from Fig. 122*b* there is

$$\left.\begin{aligned} \frac{e_1}{E_1} &= \frac{K_1'}{K_1' + C_1'} \quad \text{at } t = 0 \\ \frac{e_2}{E_2} &= \frac{K_2'}{K_2' + C_2'} \quad \text{at } t = 0 \end{aligned}\right\} \qquad (2)$$

In an ordinary transformer with either grounded or isolated neutral, the initial distributions are practically zero at the neutral, Fig. 104, and in a non-resonating transformer of conventional design this is also true. Therefore, e_1/E_1 is small in (2) and so K_1' is small compared with C_1' and likewise K_2' is small compared with C_2'.

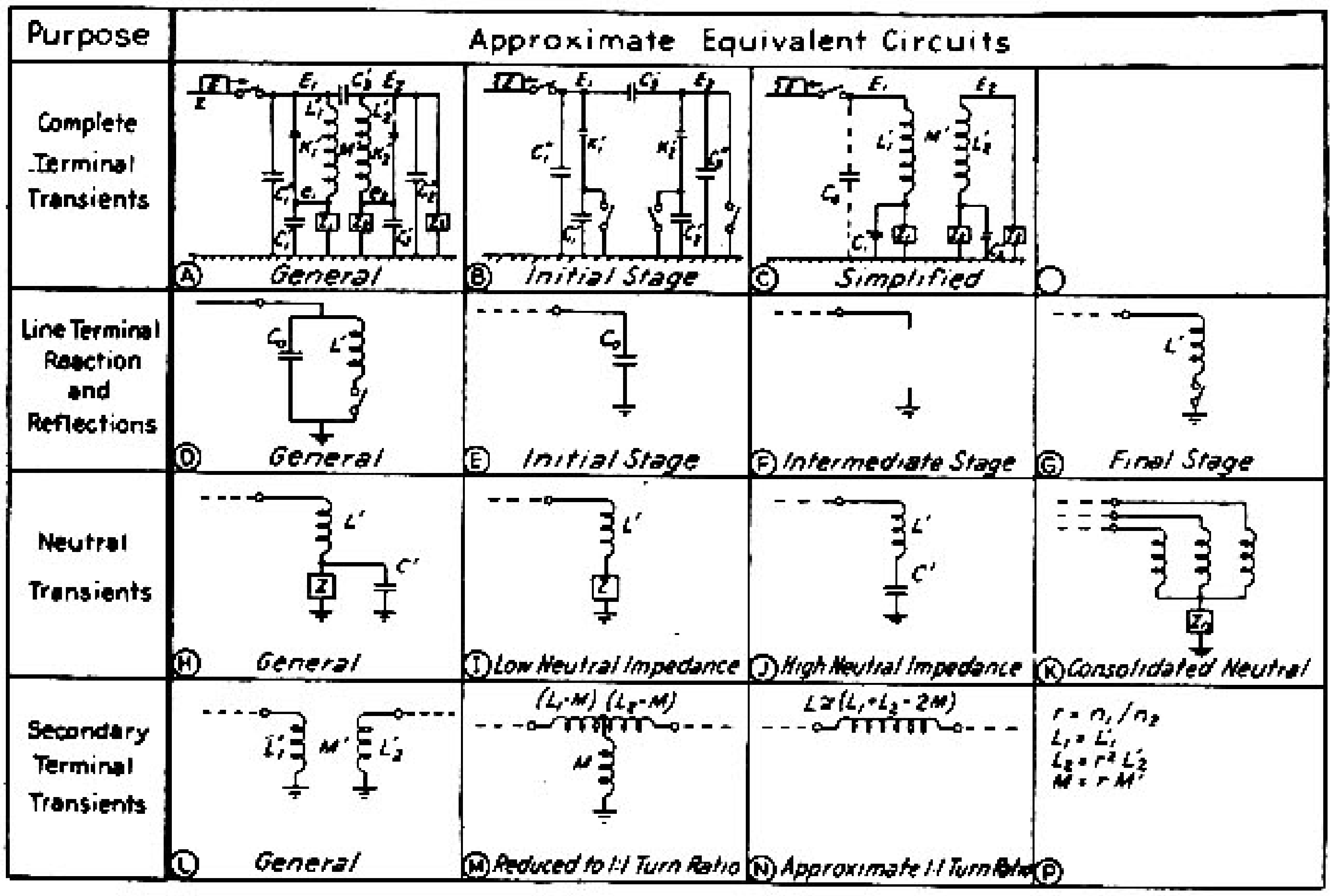


FIG. 122.—Equivalent Circuits for Terminal Transients

If $Z_2 = 0$ and $Z_3 = \infty$ and an abrupt voltage is applied to the primary line terminal, the corresponding initial voltage which appears at Z_3 can be calculated by the methods of Chapter XII, and hence in Fig. 122*b*

$$\frac{E_2}{E_1} = \frac{C_3'}{C_3' + K_2' + C_2''} \tag{3}$$

If $Z_1 = \infty$ and $Z_2 = Z_3 = 0$, the natural frequency of oscillation of the primary neutral can be calculated from the general equations of Chapter XII. Under these conditions the effective inductance of the primary becomes the leakage inductance

$$L_1'' = L_1' - (M'^2/L_2') \tag{4}$$

and the neutral voltage is

$$e_1 = E_1\left[1 - \frac{C_1'}{C_1' + K_1'}\cos\left\{\frac{t}{\sqrt{L_1''\,(C_1' + K_1')}}\right\}\right] \tag{5}$$

or since K_1' is small compared with C_1'

$$e_1 \cong E_1 \left\{1 - \cos \frac{t}{\sqrt{L_1'' C_1'}}\right\}. \tag{6}$$

and thus the neutral voltage oscillates to double the terminal voltage E_1 (as is substantially the case in the actual circuit) and at a frequency $(1/2\,\pi\sqrt{L_1'' C_1'})$. This is the same as the natural frequency of oscillation of an isolated neutral if

$$C_1' = (2\,\pi f_1)^2 L_1'' \tag{7}$$

Likewise $\quad C_2' = (2\,\pi f_2)^2 L_2''$

Hence by (2)

$$\left.\begin{aligned} K_1' &= (2\,\pi f_1)^2 L_1'' \left(\frac{e_1}{E_1 - e_1}\right) \\ K_2' &= (2\,\pi f_2)^2 L_2'' \left(\frac{e_2}{E_2 - e_2}\right) \end{aligned}\right\} \tag{8}$$

and by (3) and (1)

$$C_3' = \frac{E_2}{E_1} C_{02} \tag{9}$$

and by (1)

$$\left.\begin{aligned} C_1'' &= C_{01} - (K_1' + C_3') \\ &= C_{01} - \frac{E_2}{E_1} C_{02} - (2\,\pi f_1)^2 L_1'' \left(\frac{e_1}{E_1 - e_1}\right) \\ C_2'' &= C_{02} - (K_2' + C_3') \\ &= C_{02} - \frac{E_2}{E_1} C_{02} - (2\,\pi f_2)^2 L_2'' \left(\frac{e_2}{E_2 - e_2}\right) \end{aligned}\right\} \tag{10}$$

It is thus possible to assign rational values to all the constants of the general equivalent circuit. But this circuit is too complicated to be of any use for actual numerical calculations. The next logical step is to simplify it to meet the more restricted requirements of specific applications.

Since the capacitances K_1' and K_2' are intended merely to yield the slight initial voltages at the neutral, and since they do not materially affect either the amplitude or frequency of the neutral oscillation, it is permissible to delete them. This move is validated by cathode-ray oscillograms of the neutral transients.

The capacitance C_3' accounts for the initial voltage at the secondary line terminal. If the secondary is grounded, C_3' loses its significance and may then be deleted, provided that C_1'' and C_2'' are correspondingly increased to satisfy (1). But if the line terminal is connected to an impedance, then by (39*a*) of Chapter XII,

$$\frac{E_2}{E_1} = 0.193\ \epsilon^{-10.8t} \tag{11}$$

which shows that, although this electrostatic transient has an appreciable crest value, it is over with in a small fraction of a microsecond, and long before the electromagnetic transient has made any headway. In a non-resonating * transformer this electrostatic transient is of longer duration, having a time constant of the order of one microsecond. Of course the time constant is greater if the impedance Z_3 exceeds the 500 ohms assumed, but in any practical case it is too short to exercise much influence on the character of the secondary terminal transient. It may therefore be deleted in all cases where the secondary line terminal is connected to an underground cable, overhead line, or generator. The equivalent circuit now simplifies to that of Fig. 122*c*, in which the effective capacitance of the transformer now appears lumped between line and ground.

It is pertinent at this point to investigate the possibilities of the high-frequency internal oscillations being refracted to the terminal impedances. The oscillatory components of current transmitted to a terminal impedance must be less than those which would exist if the impedance were zero. Referring to the numerical example at the end of Chapter XII and Equation (79), the maximum oscillatory current for grounded terminals is

$$\begin{aligned}(i_{L2} + i_{K2}) &= \sum \Big[\frac{\omega}{\lambda} A\ (r\,C_2 + r\,C_3 - C_3)\sin \omega t - \frac{\Omega}{\lambda} A' \\ &\qquad (r'\,C_2 + r'\,C_3 - C_3)\sin \Omega\, t\Big] \qquad (12) \\ &\cong 30 \times 10^{-6}\,E_1\end{aligned}$$

If this current were to flow in a terminal impedance of 500 ohms it would cause a voltage of only

$$E_2 = 0.015\,E_1 \tag{13}$$

from which it is evident that the oscillatory components do not find their way appreciably into terminal impedances of the usual values.

An unexpected similarity between the equivalent circuit of Fig. 122

* See Chapter XVI.

and the complete circuit of Fig. 101 is brought to light by putting $Z_1 = Z_2 = \infty$ and $Z_3 = 0$. Then the solution of the circuit gives

$$e_1 = \frac{E_1}{(L_1' L_2' - M'^2)(C_1' C_2')}\left[\frac{1}{\omega^2 \Omega^2} + \frac{1 - \omega^2 L_2' C_2'}{\omega^2 (\omega^2 - \Omega^2)} \cos \omega t - \frac{1 - \Omega^2 L_2' C_2'}{\Omega^2 (\omega^2 - \Omega^2)} \cos \Omega t\right] \quad (14)$$

$$e_2 = \frac{E_1 M'}{(L_1' L_2' - M'^2)(\omega^2 - \Omega^2) C_2'} [\cos \omega t - \cos \Omega t] \quad (15)$$

$$\left.\begin{matrix}\omega^2 \\ \Omega^2\end{matrix}\right\} = \frac{L_1' C_1' + L_2' C_2' \pm \sqrt{(L_1' C_1' - L_2' C_2')^2 + 4 M'^2 C_1' C_2'}}{2 (L_1' L_2' - M'^2) C_1' C_2'} \quad (16)$$

which shows that there are two separate frequencies in the equivalent circuit, just as there are in the general solution, and of these two frequencies, one predominates in the primary (as it does in the general solution), and both frequencies are of practically the same importance in the secondary (as they are in the general solution).

REACTION AT LINE TERMINAL

The effective capacitance of a transformer, with respect to the line terminal, is quite small, of the order of 0.0002 microfarad to 0.001 microfarad for ordinary transformers, and from 10 to 20 or more times as much for non-resonating transformers. This effective capacitance controls the initial stage of the reflections from the line terminal, but very soon becomes fully charged. Eventually the transformer (if grounded neutral) acts as a pure inductance. These considerations suggest the equivalent circuit of Fig. 122*D* for calculating the terminal reaction, the switch in the inductive branch being open for an isolated neutral and closed for a grounded neutral. The solution for this circuit—inductance L and capacitance C in parallel at the end of a transmission line of surge impedance Z—is

$$e = \frac{(2E)}{ZC} \frac{1}{n - m} (\epsilon^{-mt} - \epsilon^{-nt}) \quad (17)$$

where

$$\left.\begin{aligned} n &= \frac{1}{2ZC} + \sqrt{\frac{1}{(2ZC)^2} - \frac{1}{(LC)}} \cong \frac{1}{ZC} \\ m &= \frac{1}{2ZC} - \sqrt{\frac{1}{(2ZC)^2} - \frac{1}{(LC)}} \cong \frac{Z}{L} \end{aligned}\right\} \quad (18)$$

Hence, approximately

$$e = (2\,E)\left(\epsilon^{-\frac{Z}{L}t} - \epsilon^{-\frac{t}{ZC}}\right) \tag{19}$$

For example, if $C = 0.001 \times 10^{-6}$, $L = 0.05$, and $Z = 500$

$$e = (2\,E)\;(\epsilon^{-0.01t} - \epsilon^{-2.0t})$$

Thus the transient consists of a steep front determined by Z and C, and a long tail determined by Z and L. Since the electrostatic transient subsides long before sufficient current to exert any appreciable influence flows through the inductance, it follows that the two stages of the transient may be considered separately; that is, the initial reflection calculated as from a pure capacitance Fig. 122*E*, and the final reflection as from a pure inductance, Fig. 122*G*.

If the applied wave is short compared with the time constant of the electromagnetic transient, or if the natural period of internal oscillations is short in comparison therewith, it is a permissible approximation to regard the transformer simply as an open circuit at the end of the line, so that incident waves double upon impact.

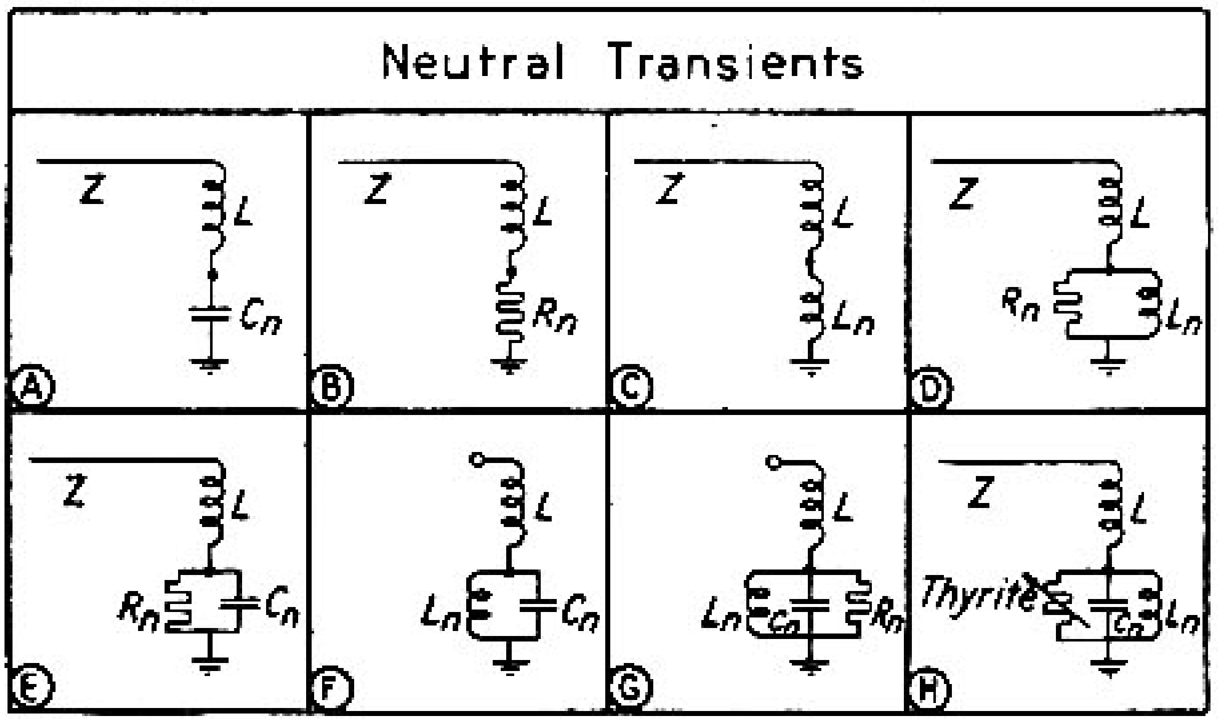
If there is a choke coil or reactor in series with the transformer, violent oscillations may occur between the series inductance and the effective capacitance of the transformer. The frequency of these oscillations is too high to penetrate the inductance of the transformer, and the transformer then behaves substantially as a capacitance. Representative calculations of the effect of choke coils and reactors in series with a transformer were given in Figs. 39 to 49 inclusive.

NEUTRAL TRANSIENTS

The simplified equivalent circuit (Fig. 122*H*) for the calculation of transformer neutral transients * was deduced from the general equivalent circuit of Fig. 122*A*. As practical special cases thereof are the circuits of Fig. 123. The equations for the neutral voltage are given in the table, with the exception of Circuit *H*, which will be solved by the step-by-step method. These transients are either exponential or oscillatory, depending upon the relative values of the circuit constants. The maximum neutral voltage in the aperiodic cases, or the axis of neutral voltage oscillation in the oscillatory cases, is plotted in Fig. 124. The equations apply for an infinite rectangular wave, or for a finite rectangular wave during its duration. Circuit *C* is fictitious, since the neutral is bound to oscillate by virtue of the capaci-

* First used by K. K. Palueff and J. H. Hagenguth. "Effect of Transient Voltage on Power Transformer Design. II," by K. K. Palueff, *A.I.E.E. Trans.*, Vol. 49.

tance of the equivalent circuit. Therefore, Circuit F is the one which really defines the neutral transient when grounded through an inductance.



	E_n	α	ω_0^2	Equation
A	$2E$	$\frac{Z}{2L}$	$\frac{1}{LC_n}$	$E_n\left[1-\frac{\omega_0}{\omega}\epsilon^{-\alpha t}\sin(\omega t+\phi)\right]$
B	$\frac{2ER_n}{Z+R_n}$	$\frac{Z+R_n}{L}$		$E_n(1-\epsilon^{-\alpha t})$
C	$\frac{2EL_n}{L+L_n}$	$\frac{Z}{L+L_n}$		$E_n\epsilon^{-\alpha t}$
D	$\frac{2E}{\Omega L}$	$\frac{R_n(L+L_n)+ZL_n}{2LL_n}$	$\frac{ZR}{LL_n}$	$E_n\frac{1}{\Omega L}\epsilon^{-\alpha t}\sinh\Omega t$
E	$\frac{2ER_n}{Z+R_n}$	$\frac{L+ZR_nC_n}{2R_nC_nL}$	$\frac{R_n+Z}{R_nC_nL}$	$E_n\left[1-\frac{\omega_0}{\omega}\epsilon^{-\alpha t}\sin(\omega t+\phi)\right]$
F	$\frac{2EL_n}{L+L_n}$	0	$\frac{L+L_n}{LL_nC_n}$	$E_n(1-\cos\omega t)$
G	$\frac{2EL_n}{L+L_n}$	$\frac{1}{C_nR_n}$	$\frac{L+L_n}{LL_nC_n}$	$E_n\left[1-\frac{\omega_0}{\omega}\epsilon^{-\alpha t}\sin(\omega t+\phi\right]$
H				see Text

$\omega^2 = (\omega_0^2-\alpha^2)$, $\tan\varphi = \frac{\omega}{\alpha}$, $\Omega^2 = (\alpha^2-\omega_0^2)$

Fig. 123.—Neutral Transients

If the neutral is to be held below a specified voltage E_0, it is sufficient to meet the following conditions:

$$\begin{array}{ll}\text{Oscillatory circuit} & \left\{\begin{array}{l} L_n/L \text{ from Fig. 124} \\ \text{using } E_n = 2\,E_0 \end{array}\right. \\ \text{No damping} & \end{array}$$

$$\begin{array}{ll}\text{Oscillatory circuit} & \left\{\begin{array}{l} R_n/Z \text{ from Fig. 124, using } E_n = E_0 \\ \alpha^2 \geq \omega_0^2 \end{array}\right. \\ \text{Arbitrary damping} & \end{array}$$

$$\text{Aperiodic circuit} \quad \{\ R_n/Z \text{ from Fig. 124, } E_n = E_0$$

For example, if a transformer bank has a leakage inductance of 0.0476 henry, and the neutrals are grounded through a common reactor of 0.573 henry, the neutral may oscillate to a value of 1.85 ($2E$). Suppose that it is required to limit the neutral voltage to 0.50 ($2E$) by substituting a resistor, as in Circuit B. From Fig. 124, corresponding to $E_n/2E = 0.5$ there is

$$\frac{R_n}{Z} = 1 \quad \text{or} \quad R_n = 500 \text{ ohms}$$

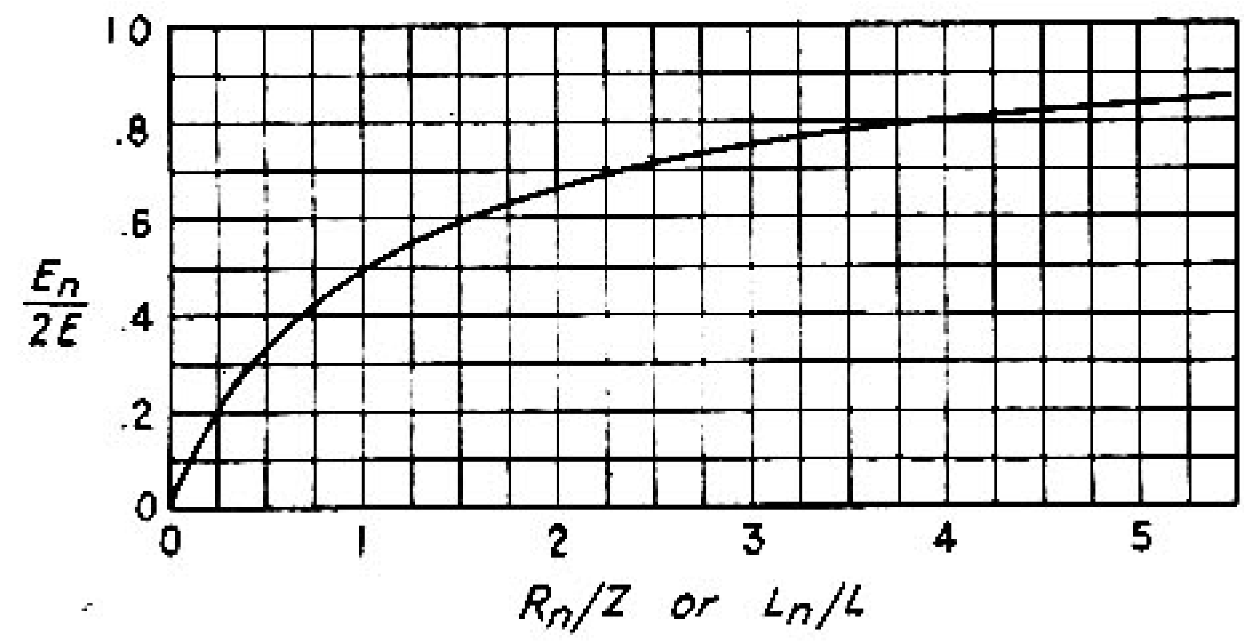


FIG. 124.—Maximum Neutral Voltage or Axis of Oscillation

Step-by-Step Method of Calculation.—From traveling-wave theory, the differential equations of the equivalent circuit, Fig. 123*H*, are

$$\left.\begin{aligned}
2e &= L_T \frac{di}{dt} + e_n + zi \\
e_n &= L \frac{di_L}{dt} \\
i_C &= C \frac{de_n}{dt} \\
i_R &= f(e_n) \text{ from the Thyrite characteristic, Fig. 36}
\end{aligned}\right\} \qquad (20)$$

Unfortunately, however, these differential equations can not be solved, on account of the non-linear relationship between i_R and e_n. It is therefore necessary to resort to a step-by-step method of successive approximations. This is accomplished by replacing the differentials with increments and rearranging the equations in their order of calculation.

$$\left.\begin{aligned}
\Delta i &= \left(\frac{2\,e - e_n - zi}{L_T}\right)\Delta t && = \text{increment in transformer current}\\
i &= \textstyle\sum \Delta i && = \text{transformer current}\\
\Delta i_L &= \frac{e_n}{L}\,\Delta t && = \text{increment in reactor current}\\
i_L &= \textstyle\sum \Delta i_L && = \text{reactor current}\\
i_R &= f\,(e_n) && = \text{resistor current}\\
i_C &= i - i_L - i_R && = \text{capacitor current}\\
\Delta e_n &= \frac{i_C}{C}\,\Delta t && = \text{increment in neutral voltage}\\
e_n &= \textstyle\sum \Delta e_n && = \text{neutral voltage}
\end{aligned}\right\} \quad (21)$$

In the above equations the increments corresponding to the increment Δt are based on the *average* values of the variables over that interval. But since the variables are not known, except at the beginning of the particular interval Δt, it is necessary to make trial assumptions as to their average values over Δt. These assumptions are greatly expedited by plotting curves of e_n and zi as the calculation progresses, not only to avoid gross errors, but also because, by extrapolation of the curves by inspection, fairly accurate guesses can be made as to the average values of e_n and zi to use over the subsequent interval. For instance, referring to Fig. 125, suppose that the calculations have progressed to the point P. By extrapolation of the curves of e_n and zi, trial average values of e_n and zi over the interval Δt may be obtained for use in the equations, and there corresponds an approximate value δe_n of the increment. But the next to the last item computed in the sequence of calculations is $\Delta e_n = (i_C/C)\ \Delta t$. If this calculated value of the increment does not check the value estimated by the free-hand extrapolation of the curve for e_n, then it means that the trial was in error and that it should be corrected. It is worth while to plot Δe_n against δe_n as shown in Fig. 125. The point where this curve crosses the 45° line defines the true value of Δe_n. Ordinarily the average between Δe_n and δe_n will prove sufficiently accurate, unless e_n has reached its "knee," where several trials may have to be made to realize precision. Of course, the accuracy depends upon how small the arbitrary intervals Δt are taken. Sometimes spurious oscillations are introduced when the increments are taken too large, Fig. 126. If such oscillations are suspected they can be detected by taking smaller increments. They are liable to occur at abrupt changes in curvature, that is, at the "knee" of the curve.

Perhaps the most efficient arrangement of the tabular schedule of

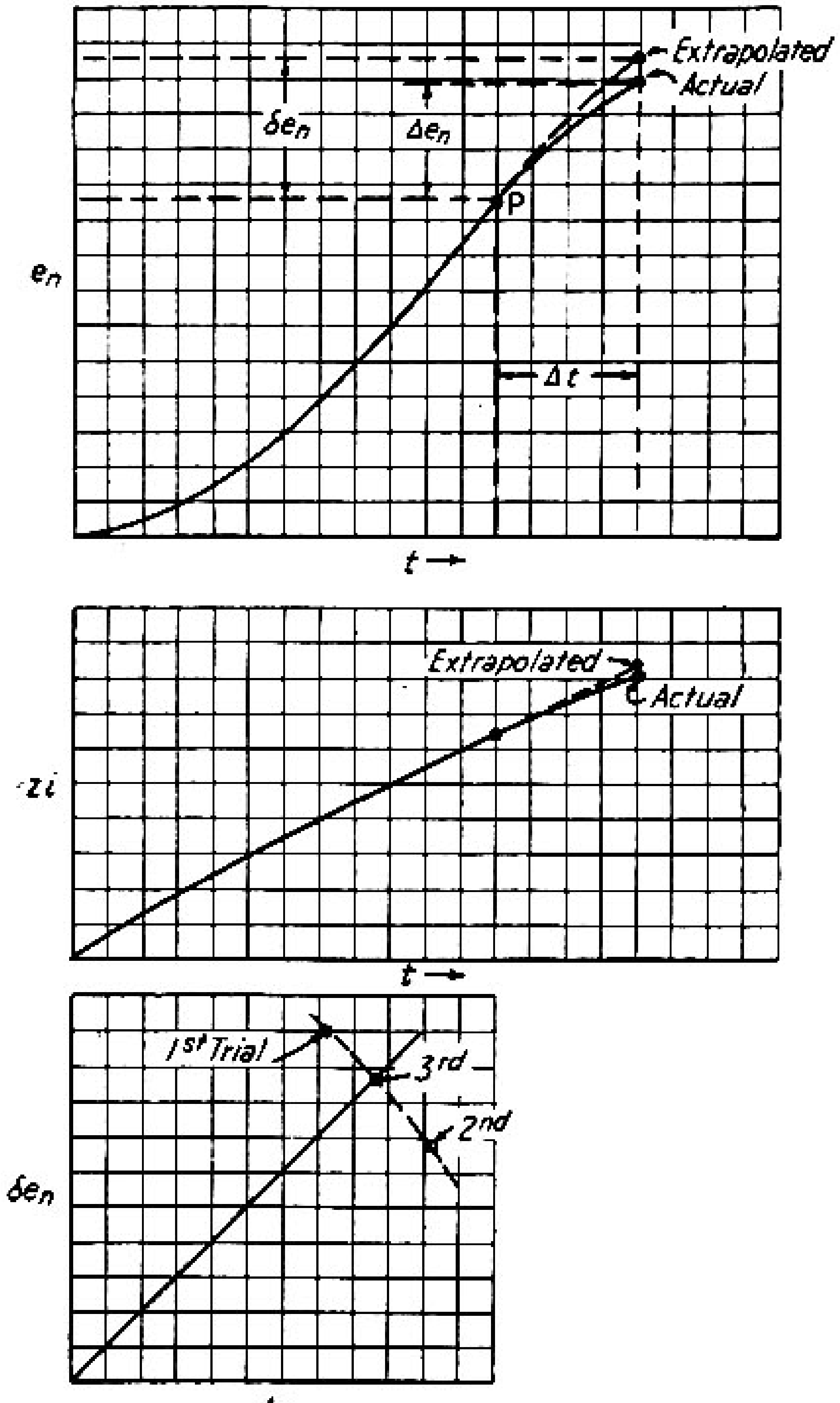


FIG. 125.—Trial Extrapolations and Their Check

calculations is that given below, in which the numerals in the body of the table indicate the order of calculation.

Δt	t	Σe	Δi	i	zi	Δi_L	i_L	i_R	i_C	Δe_n	e_n
(1)	....	....	(4)	....	....	(7)	....	....	....	(11)	....
....	(2)	(3)	....	(5)	(6)	....	(8)	(9)	(10)	....	(12)

In starting the table, there is, of course, no curve which can be extrapolated by inspection, but the initial flow of current over the first time increment is very small, and the neutral voltage will stay at practically zero potential, so that for that first increment

$$\Delta i = \frac{2\,e}{L_T}\,\Delta t \cong i_C \tag{22}$$

As a matter of fact, the influence of the reactor L is quite insignificant, and may be ignored in the calculations. Moreover, if the calcu-

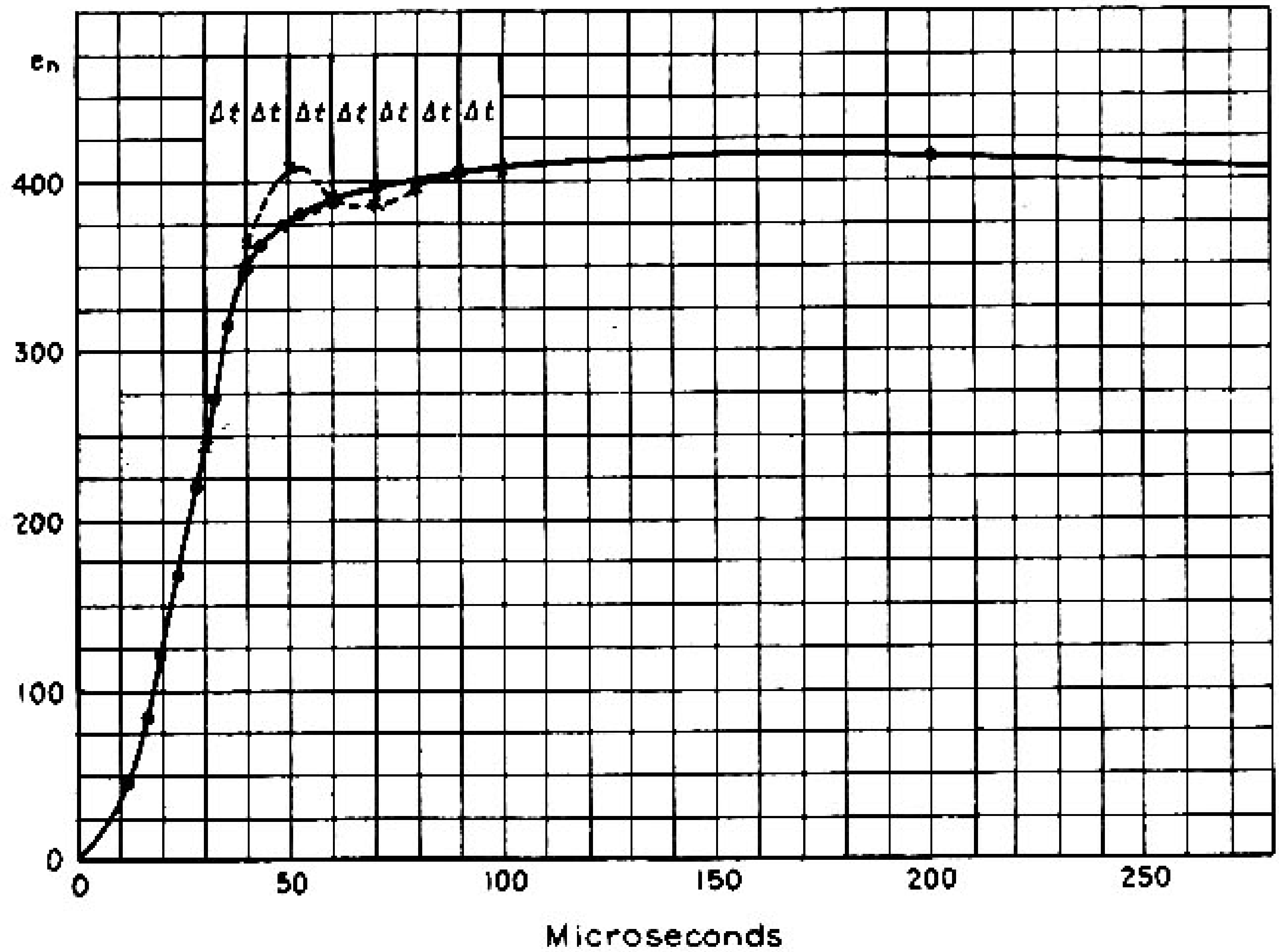


FIG. 126.—Spurious Oscillations Caused by Taking the Increments too Large.

lation is carried out merely to determine the maximum neutral voltage, and the rate of rise of the neutral voltage is of no concern, then it is permissible to ignore the capacitance C as well as L, and the equations simplify to:

$$\left.\begin{aligned} \Delta i &= \left(\frac{2\,e - e_n - zi}{L_T}\right)\Delta t \\ i &= \Sigma\Delta i \\ e_n &= f\,(i) \end{aligned}\right\} \tag{23}$$

The corresponding tabular schedule is:

Δt	t	$2e$	Δi	i	zi	e_n
(1)	....	....	(4)	....	....	....
....	(2)	(3)	....	(5)	(6)	(7)

Even the surge impedance drop zi may be neglected without affecting the result by more than a few per cent, provided that the applied wave reaches its crest within a few microseconds and has a rapidly falling tail.

Of course, for very long waves the inductance L_T comes to act as a short circuit, and the voltage across the Thyrite is determined by its own resistance and the surge impedance of the transmission line. In that case

$$2\,e = zi + e_n \tag{24}$$

The value of e_n at any instant of time is readily found by plotting a curve of $(e_n + zi)$ against i, and the point at which this curve reaches a value equal to the value of $2\,e$ at that instant defines the current i and the corresponding neutral voltage e_n from the Thyrite characteristic. However, such a calculation should be employed only to determine an upper limit to the neutral voltage, which will not actually be reached—in other words, it provides a conservative estimate.

Separate Effect of the Neutral Impedor Elements.—In Fig. 127 are shown calculated curves of the neutral voltage for L and C only

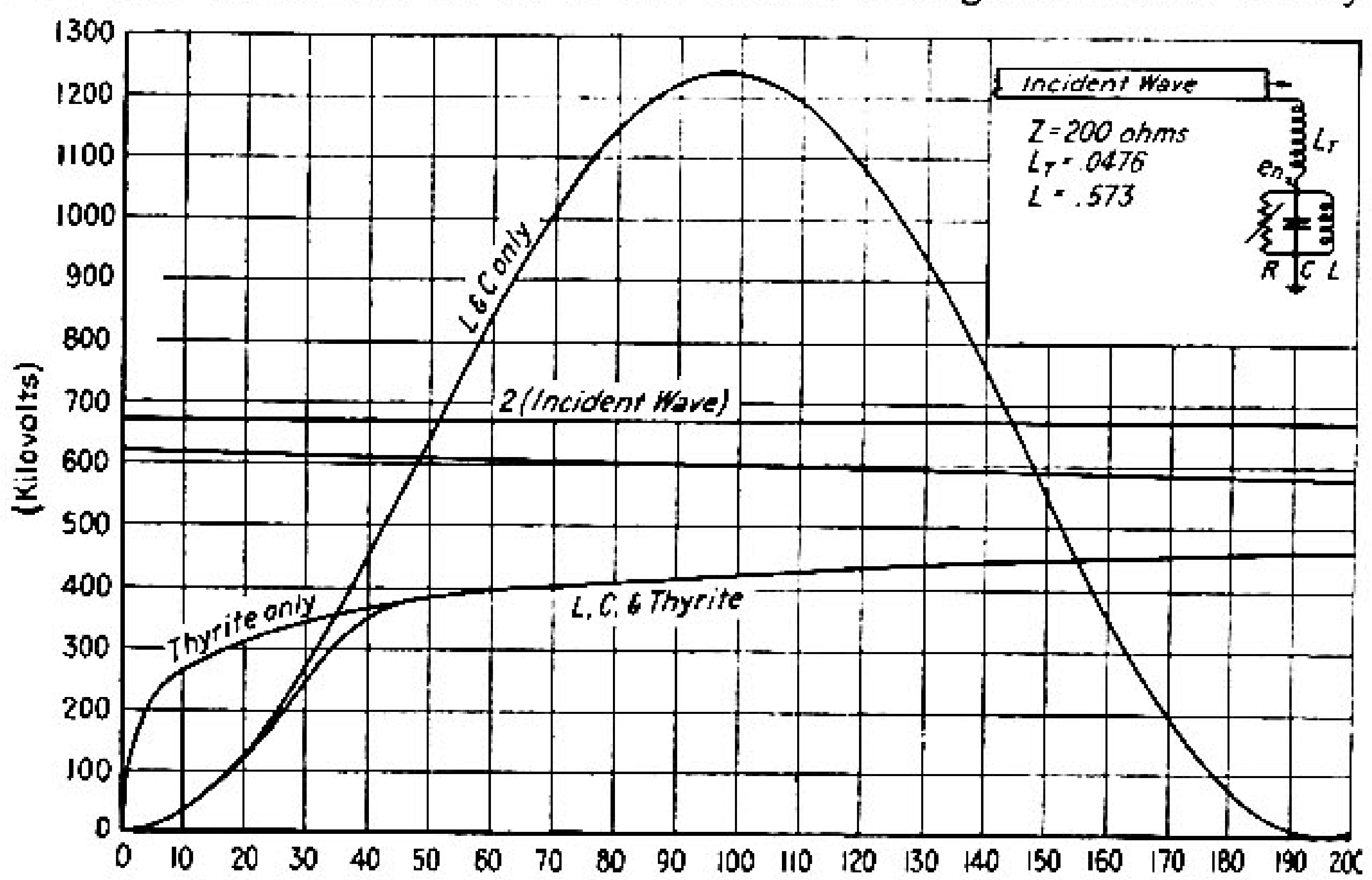


FIG. 127.—Effect of the Neutral Impedance Elements

L and Thyrite only, and L, C, and Thyrite in combination, for an infinite rectangular applied wave. When only the capacitance and the reactor are used, the neutral voltage is a slightly damped sinusoidal oscillation, or, neglecting the damping due to the surge impedance, it is

$$e_n = \frac{EL}{L + L_T}\left[1 - \cos\sqrt{\frac{L + L_T}{L\,L_T C}}\,t\right] = \frac{EL}{L + L_T}(1 - \cos \omega t) \qquad (25)$$

Now since L is large compared to L_T it is evident that e_n reaches a value nearly double the line terminal voltage E. The larger C the lower the frequency of oscillation and the slower the rate of rise of the neutral voltage. The time required for the neutral voltage to reach a particular value e_n is

$$t = \sqrt{\frac{L\,L_T C}{L + L_T}}\cos^{-1}\left[1 - \frac{L + L_T}{L}\frac{e_n}{E}\right] \qquad (26)$$

Had the applied wave been of finite length τ, then

$$e_n = \frac{EL}{L + L_T}(1 - \cos \omega t) \text{ for } t \leq \tau \qquad (27)$$

$$= \frac{2\,EL}{L + L_T}\sin\frac{\omega\tau}{2}\sin\omega\left(t - \frac{\tau}{2}\right) \text{ for } t \geq \tau \qquad (27a)$$

Thus, for finite waves whose equivalent length τ (finite rectangular wave) is less than half the period of oscillation of the neutral, the neutral voltage will not reach the upper limit of twice the terminal voltage.

When only Thyrite and L are present in the neutral impedor, the voltage, Fig. 127, rises rather abruptly at first and thereafter at a more gradual rate. The "Thyrite only" curve was calculated ignoring the presence of the transformer equivalent circuit capacitance (which is quite distinct from the auxiliary neutral impedor capacitance), and therefore the actual rise of the neutral voltage with Thyrite only would be slower than shown.

Now when all three elements are combined in the impedor, the neutral voltage first follows the C–L curve, but as the voltage across the Thyrite increases it takes a larger and larger proportion of the total current, so that the neutral voltage curve finally leaves the C–L curve and merges into the Thyrite curve. It is therefore seen that the *rate of rise* of the neutral voltage is controlled principally by the capacitor, but the *magnitude* of the neutral voltage is determined by the Thyrite.

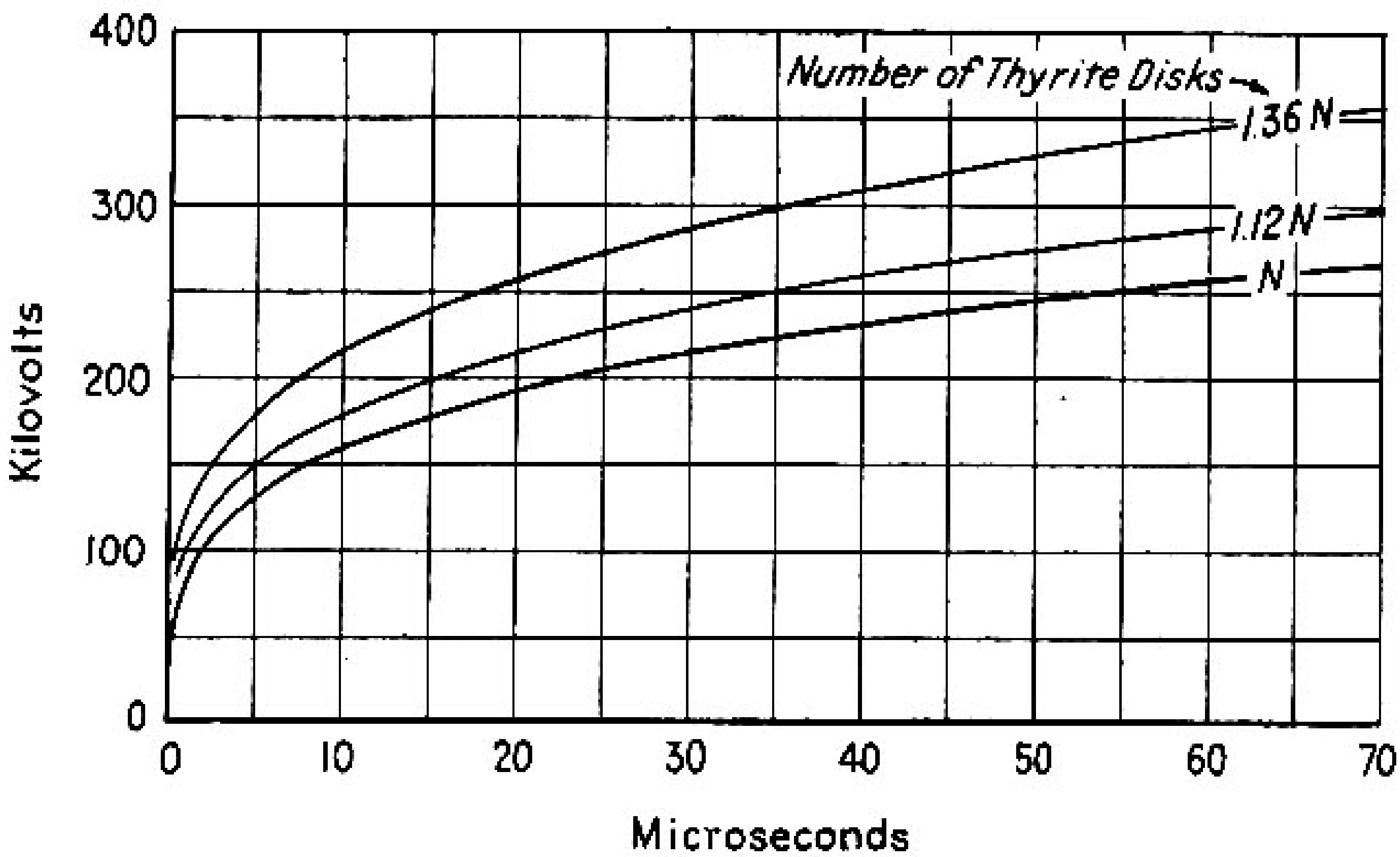


FIG. 128.—Effect of the Number of Thyrite Disks in the Neutral Impedor

The larger the number of Thyrite disks, the higher the rise of the neutral voltage. This point is illustrated by Fig. 128.

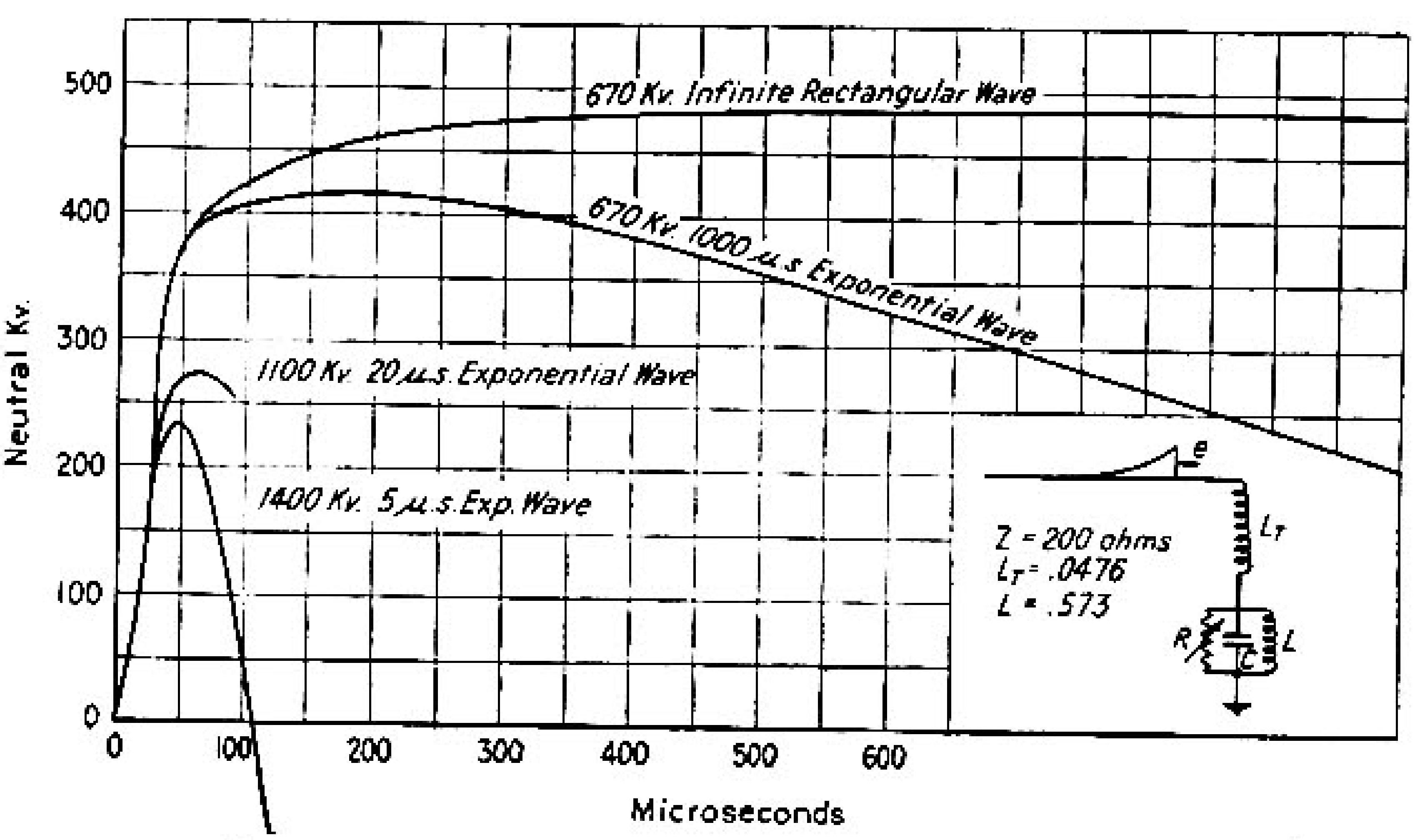


FIG. 129.—Effect of Applied Wave on the Neutral Transient

Effect of the Amplitude and Length of the Applied Wave.—Fig. 129 shows the shape and magnitude of the neutral voltage corresponding to different applied waves. The corresponding values are

Applied Wave		Neutral Voltage	
1400 kv.	5-ms. exponential	233 kv.	86-ms.
1100 kv.	20-ms. exponential	273 kv.	
670 kv.	1000-ms. exponential	416 kv.	1000-ms
670 kv.	Infinite rectangular	485 kv.	Infinite

Thus, in this particular case, a 1000-ms. wave half the amplitude of the 5-ms. wave causes nearly double the neutral voltage. In general, it is the longer, relatively low-voltage waves which cause the greatest distress at the neutral, not only because the magnitude of the neutral voltage is greater for these long waves, but also because it persists for a much longer time. It is interesting to note that the neutral voltage may be maintained for a long time after the applied wave has vanished. For instance, the 5-ms. wave gives rise to a neutral voltage which does not reach its maximum until 80 ms., and it is 107 ms. until it passes through zero, where it reverses polarity and commences the second loop of its oscillation.

SECONDARY TERMINAL TRANSIENT

It has been shown that the general equivalent circuit reduces to that of Fig. 122*L* when the terminals are connected to overhead lines, cables, or generators. This is the circuit corresponding to Equations (49) and (50) of Chapter XII. It was first employed for a comprehensive study of the transfer of waves from the primary to the secondary circuit by K. K. Palueff and J. H. Hagenguth.* The review given here is taken, with minor changes, from their paper.

The equations of the circuit of Fig. 122*L* when connected to a primary surge impedance Z_1' and a secondary surge impedance Z_2' are

$$\left.\begin{aligned} 2\,E &= (Z_1' + p\,L_1')\,I_1 + p\,M'I_2 \\ 0 &= (Z_2' + p\,L_2')\,I_2 + p\,M'I_1 \end{aligned}\right\} \tag{28}$$

Solving these two simultaneous equations, there results

$$e_2 = Z_2'I_2 = \frac{2\,E\,M'\,Z_2'\,(\epsilon^{-mt} - \epsilon^{-nt})}{\sqrt{(L_1'Z_2' + L_2'Z_1')^2 - 4\,Z_1'Z_2'\,(L_1'L_2' - M'^2)}}$$
$$= A\,(\epsilon^{-mt} - \epsilon^{-nt}) = A\,\epsilon^{-mt}\,[1 - \epsilon^{-(n-m)t}] \tag{29}$$

* "Effect of Transient Voltages on Power Transformer Design. IV," *A.I.E.E. Trans.*, Vol. 51, 1932.

where

$$\left.\begin{matrix} n \\ m \end{matrix}\right\} = \frac{\left\{\begin{matrix} (L_2'Z_1' + L_1'Z_2') \\ \pm\sqrt{(L_2'Z_1' + L_1'Z_2')^2 - 4\,Z_1'Z_2'\,(L_1'L_2' - M'^2)} \end{matrix}\right\}}{2\,(L_1'\,L_2' - M'^2)} \tag{30}$$

where the plus sign belongs to n and the minus sign to m.

Reducing Fig. 122L to a 1 : 1 turn ratio by the usual substitutions

$$Z_1 = Z_1' \qquad r = n_1\ n_2 \qquad Z_2 = r^2\,Z_2'$$

$$L_1 = L_1' \qquad M = r\,M' \qquad L_2 = r^2\,L_2'$$

there results the equivalent circuit (Fig. 122M) in which

$$L = \frac{L_1\,L_2 - M^2}{L_2} \cong L_1 + L_2 - 2\,M = \text{leakage inductance}$$

Moreover, without appreciable error,

$$L_1 \cong L_2 \cong M$$

Hereby, rearranging and expanding (30) by the binomial theorem

$$m = \frac{L_2\,Z_1 + L_1\,Z_2}{2\,L_2\,L}\left[1 - \sqrt{1 - \frac{4\,Z_1\,Z_2\,L_2\,L}{(L_2\,Z_1 + L_1\,Z_2)^2}}\right]$$

$$= \frac{L_2\,Z_1 + L_1\,Z_2}{2\,L_2\,L}\left[1 - \left\{1 - \frac{2\,Z_1\,Z_2\,L_2\,L}{(L_2\,Z_1 + L_1\,Z_2)^2} + \ldots\right\}\right]$$

$$\cong \frac{Z_1\,Z_2}{L_2\,Z_1 + L_1\,Z_2} \cong \frac{Z_1\,Z_2}{L_1\,(Z_1 + Z_2)} \tag{31}$$

$$n - m = \frac{L_2\,Z_1 + L_1\,Z_2}{L_2\,L}\sqrt{1 - \frac{4\,Z_1\,Z_2\,L_2\,L}{(L_2\,Z_1 + L_1\,Z_2)^2}}$$

$$\cong \frac{L_2\,Z_1 + L_1\,Z_2}{L_2\,L} \cong \frac{Z_1 + Z_2}{L} \tag{32}$$

$$A \cong \frac{2\,E}{r}\,\frac{Z_2}{Z_1 + Z_2} \tag{33}$$

Thus (29) may be considered as the *product* of two transients—the comparatively slow transient ϵ^{-mt} in which m is the ratio of Z_1 and Z_2 in parallel, to the self-inductance L_1; and the much faster transient $[1 - \epsilon^{-(n-m)t}]$ in which $(n - m)$ is the ratio of Z_1 and Z_2 in series, to the leakage inductance L. The relative period of these two transients is defined by the ratio of their time constants

$$\frac{\text{Time of } L_1 \text{ transient}}{\text{Time of } L \text{ transient}} = \frac{L_1 (Z_1 + Z_2)^2}{L Z_1 Z_2} \tag{34}$$

which is of the order of several hundred. Therefore, the crest and front of the secondary transient are practically independent of the L_1 transient, and for ordinary purposes the L_1 transient may be ignored entirely, so that successive approximations are

$$\begin{aligned} e_2 &= A\, \epsilon^{-mt} [1 - \epsilon^{-(n-m)t}] \\ &\cong \frac{2E}{r} \frac{Z_2}{Z_1 + Z_2} \epsilon^{-\frac{Z_1 Z_2}{Z_1 + Z_2} \frac{t}{L_1}} \left[1 - \epsilon^{-\frac{Z_1 + Z_2}{L} t}\right] \\ &\cong \frac{2E}{r} \frac{Z_2}{Z_1 + Z_2} \left[1 - \epsilon^{-\frac{Z_1 + Z_2}{L} t}\right] \end{aligned} \tag{35}$$

This latter approximation is the equivalent circuit (Fig. 122N). The maximum is

$$E_2 = \frac{2E}{r} \frac{Z_2}{Z_1 + Z_2} = \frac{2E}{r} \frac{r^2 Z_2'}{Z_1' + r^2 Z_2'} \tag{36}$$

and the front of the wave (to 95 per cent of crest) is

$$T_2 = \frac{3L}{Z_1 + Z_2} = \frac{3L}{Z_1' + r^2 Z_2'} \tag{37}$$

It is evident, then, that, if $r^2 Z_2'$ is large compared with Z_1,

$$E_2 \cong \frac{2E}{r} = (2E) \frac{n_2}{n_1} \tag{38}$$

$$T_2 \cong \frac{3L}{r^2 Z_2'} \tag{39}$$

The larger L and the smaller r and Z_2', the slower the front. If there is a neutral resistance R_n it may be included with Z_1' in (36) and (37) and is thus seen to decrease E_2 and T_2. If there is a neutral reactor L_n it may be included in L and is thus seen to increase the front.

The above equations hold for infinite rectangular waves. For waves of any other shape Duhamel's theorem is available. In particular, if the incident wave is

$$E\,(\epsilon^{-at} - \epsilon^{-bt}) \tag{40}$$

Equation (29) becomes

$$e_2 = 2\,E\,A\left[\left(\frac{a}{m-a} - \frac{b}{m-b}\right)\epsilon^{-mt} - \left(\frac{a}{n-a} - \frac{b}{n-b}\right)\epsilon^{-nt} + \left(\frac{a}{n-a} - \frac{a}{m-a}\right)\epsilon^{-at} - \left(\frac{b}{n-b} - \frac{b}{m-b}\right)\epsilon^{-bt}\right] \tag{41}$$

Repeated Reflections.—If the transformer secondary is connected to the surge impedance Z_3 of the generator through a short length of cable of surge impedance Z_2, then

$$R_g = \frac{Z_3 - Z_2}{Z_3 + Z_2} = \text{reflection operator at generator}$$

$$R_T = \frac{p\,L + Z_1 - Z_2}{p\,L + Z_1 + Z_2} = \frac{p + (Z_1 - Z_2)/L}{p + (Z_1 + Z_2)/L}$$

$$= \frac{p + \beta}{p + \alpha} = \text{reflection operator at transformer}$$

$$e_2 = A\,(1 - \epsilon^{-\alpha t}) = \text{initial transmitted wave}$$

and if T is the time of transit of a wave on the cable, the resultant voltage is

$$E_2 = e_2 + (R_g + R_g R_T)\,e_2(t - 2\,T) + (R_g^2 R_T + R_g^2 R_T^2)\,e_2(t - 4\,T) + (R_g^3 R_T^2 + R_g^3 R_T^3)\,e_2\,(t - 6\,T) + \ldots \tag{42}$$

The general term (see appendix to this chapter) is

$$\begin{aligned}
R_g^n R_T^n e_2 &= A\left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \left(\frac{p+\beta}{p+\alpha}\right)^n (1 - \epsilon^{-\alpha t}) \\
&= A\left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \left[\left(\frac{p+\beta}{p+\alpha}\right)^n - \epsilon^{-\alpha t}\left(\frac{p+\beta-\alpha}{p}\right)^n\right] \\
&= A\left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \left\{\frac{\beta^n}{\alpha^n} - \epsilon^{-\alpha t}\left[\frac{(\beta-\alpha)^n t^n}{\lfloor n} \right.\right. \\
&\quad + \sum_{k=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)^k \frac{(\beta\,t)^{n-k}}{\lfloor n-k} + \frac{\lfloor n\,(\beta-\alpha)^{n-k}\,t^{n-k}}{\lfloor k\,\lfloor n-k)\,\lfloor n-k} \right. \\
&\quad \left.\left.\left. - \frac{\lfloor n\ \beta^{n-k}}{\lfloor k\,\lfloor n-k} \sum_{r=1}^{k} \frac{\lfloor k-1\,(-\alpha)^{k-r}\,t^{n-r}}{\lfloor r-1\,\lfloor k-r\ \ \lfloor n-r}\right\}\right]\right\}
\end{aligned} \tag{43}$$

A slightly different form for the above equation can be obtained by substituting

$$(1 - \epsilon^{-\alpha t}) = \frac{\alpha}{p + \alpha}$$

whereupon the operator evaluates to

$$R_g^n\, R_T^n\, e_2 = \alpha\, A \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \frac{(p + \beta)^n}{(p + \alpha)^{n+1}}$$

$$= A \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \left\{\left(\frac{\beta}{\alpha}\right)^n - \epsilon^{-\alpha t}\left[\frac{(\beta\, t)^n}{n!} + \sum_1^n \left\{\left(\frac{\beta}{\alpha}\right)^k \frac{(\beta\, t)^{n-k}}{(n-k)!} + \frac{n!\, \beta^{n-k}}{k!\,(n-k)!} \sum_1^k \frac{(-\alpha)^{k-r+1}\, t^{n-r+1}}{(k-r)!\;(n-r+1)!\;(r-1)!}\right\}\right]\right\} \tag{44}$$

If the applied wave is $E\, \epsilon^{-at}$ instead of infinite rectangular, the operator becomes

$$R_g^n\, R_T^n\, e_2 = \alpha\, A \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \frac{(p + \beta)^n}{(p + \alpha)^{n+1}}\, \epsilon^{-at}$$

$$= \alpha\, A\, \epsilon^{-at} \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^n \frac{(p + \beta - a)^n}{(p + \alpha - a)^{n+1}} \tag{45}$$

The solution is then the same as (44) multiplied by ϵ^{-at}, in which α is replaced by $(\alpha - a)$ and β is replaced by $(\beta - a)$.

The first two reflections in (43) are:
If $n = 1$

$$R_g\, R_T\, e_2 = A \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right) \left\{\frac{\beta}{\alpha}(1 - \epsilon^{-\alpha t}) - \left(\frac{\beta}{\alpha} - 1\right) \alpha\, t\, \epsilon^{-\alpha t}\right\}$$

If $n = 2$

$$R_g^2\, R_T^2\, e_2 = A \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right)^2 \left\{\left(\frac{\beta}{\alpha}\right)^2 - \frac{\epsilon^{-\alpha t}}{2}\left[(\alpha\, t)^2 \left(\frac{\beta}{\alpha} - 1\right)^2 + 2\, \alpha\, t \left(\frac{\beta^2}{\alpha^2} - 1\right) + 2\left(\frac{\beta}{\alpha}\right)^2\right]\right\}$$

Three-Phase Banks.—The foregoing analysis may be adapted to the calculation of three-phase banks of transformers subjected to the impact of traveling waves on one, two, or three line conductors. Consider the delta-delta bank of Fig. 130 with a wave E arriving on primary phase A. From conditions of symmetry the division of current is known, and therefrom

$$2\,E = Z_1\,I + p\,L\frac{I}{2} + Z_1\frac{I}{2} + Z_2\frac{I}{2} + Z_2\,I$$

$$= [(Z_1 + Z_2)\,1.5 + 0.5\,p\,L)]\,I \tag{46}$$

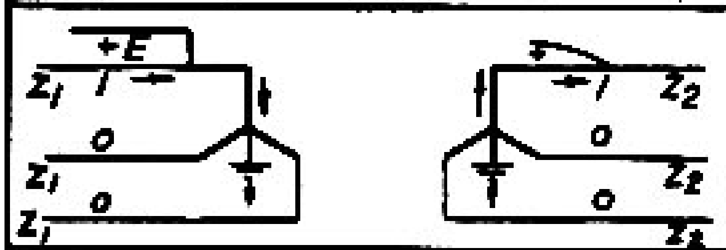
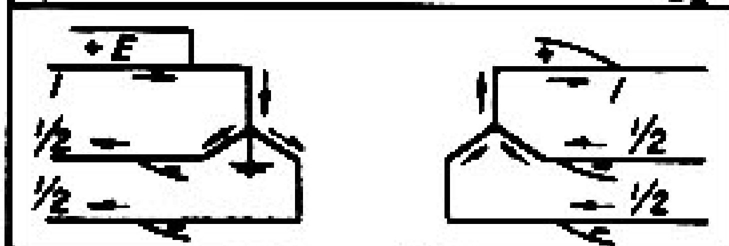
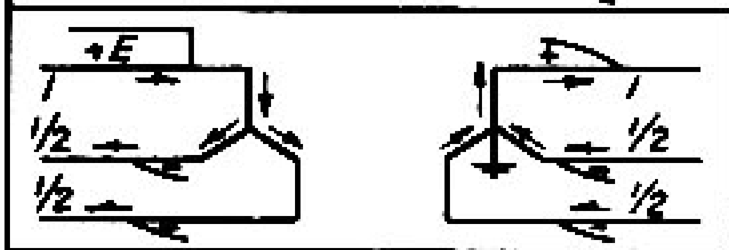
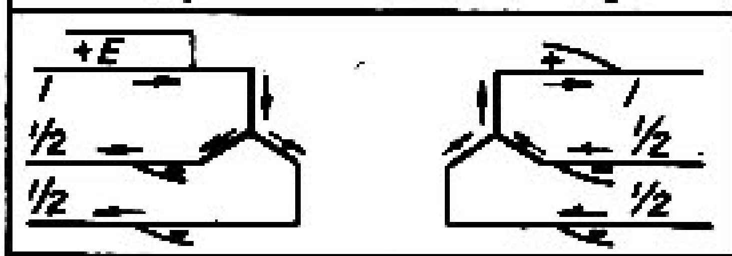
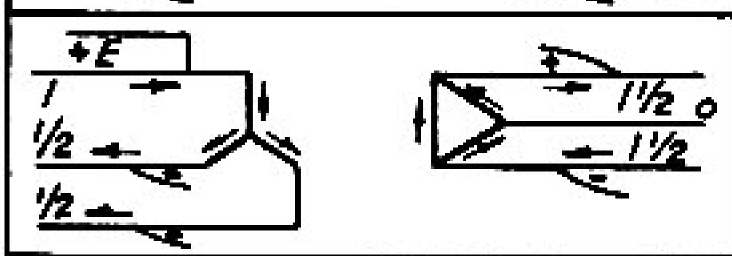
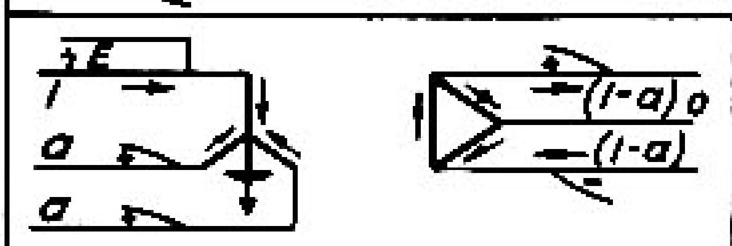
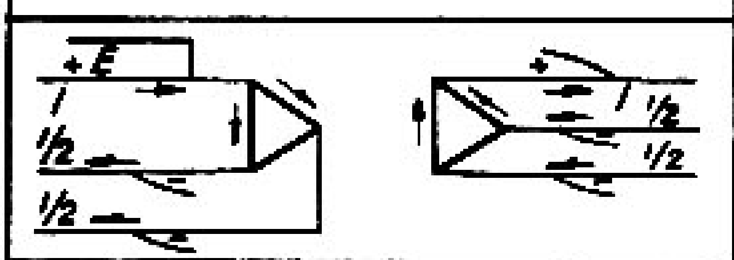
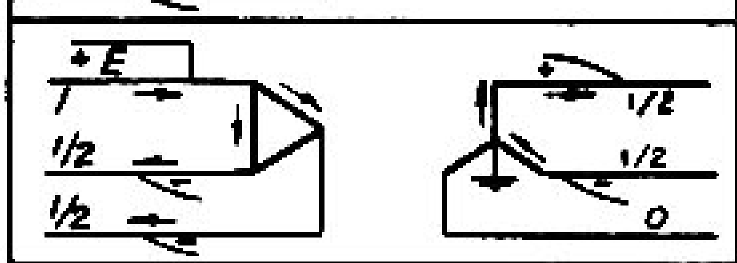
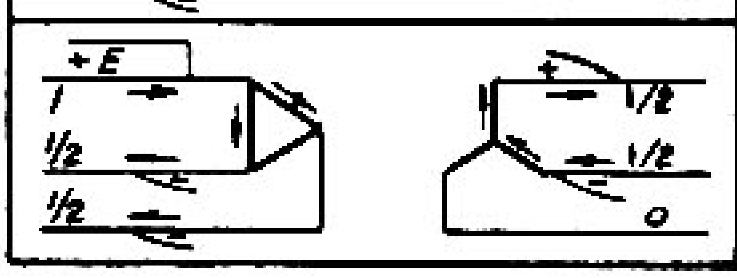
BANK CONNECTION Primary / Secondary	RELATIVE Crest	RELATIVE Front	CREST FACTOR	TIME CONSTANT
	1	1	$\frac{2E}{r}\,\frac{Z_2}{Z_1+Z_2}$	$\frac{L}{Z_1+Z_2}$
	$\frac{2}{3}$	1	$\frac{2E}{r}\,\frac{2}{3}\,\frac{Z_2}{Z_1+Z_2}$	$\frac{L}{Z_1+Z_2}$
	$\frac{2}{3}$	1	$\frac{2E}{r}\,\frac{2}{3}\,\frac{Z_2}{Z_1+Z_2}$	$\frac{L}{Z_1+Z_2}$
	$\frac{2}{3}$	1	$\frac{2E}{r}\,\frac{2}{3}\,\frac{Z_2}{Z_1+Z_2}$	$\frac{L}{Z_1+Z_2}$
	$\frac{Z_1+Z_2}{Z_1+3Z_2}$	$\frac{Z_1+Z_2}{Z_1+3Z_2}$	$\frac{2E}{r}\,\frac{Z_2}{Z_1+3Z_2}$	$\frac{L}{Z_1+3Z_2}$
	$\frac{Z_1+Z_2}{Z_1+3Z_2}$	$\frac{Z_1+Z_2}{Z_1+3Z_2}$	$\frac{2E}{r}\,\frac{Z_2}{Z_1+3Z_2}$	$\frac{L}{Z_1+3Z_2}$
	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2E}{r}\,\frac{2}{3}\,\frac{Z_2}{Z_1+Z_2}$	$\frac{L}{3(Z_1+Z_2)}$
	$\frac{Z_1+Z_2}{3Z_1+Z_2}$	$\frac{Z_1+Z_2}{3Z_1+Z_2}$	$\frac{2E}{r}\,\frac{Z_2}{3Z_1+Z_2}$	$\frac{L}{3Z_1+Z_2}$
	$\frac{Z_1+Z_2}{3Z_1+Z_2}$	$\frac{Z_1+Z_2}{3Z_1+Z_2}$	$\frac{2E}{r}\,\frac{Z_2}{3Z_1+Z_2}$	$\frac{L}{3Z_1+Z_2}$

FIG. 130.—Three-Phase Bank Connections

Therefore

$$e_2 = \frac{Z_2}{r} I = \frac{2\,E}{r}\,\frac{2\,Z_2}{L}\,\frac{1}{[p + 3\,(Z_2 + Z_1)\;L]}$$

$$= \frac{2\,E}{r}\,\frac{2}{3}\,\frac{Z_2}{Z_1 + Z_2}\left[1 - \epsilon^{-\frac{3(Z_1+Z_2)t}{L}}\right] \tag{47}$$

As a second example take the grounded neutral wye-delta bank shown in Fig. 130. In this case the division of current is not evident, so assume that a fraction a flows in the two phases not struck, and write the equation

$$2\,E = [Z_1 + p\,L - a\,Z_1 - a\,p\,L + (1 - a)\,2\,Z_2 + (1 - a)\;Z_2]I$$

$$= (1 - a)\;(p\,L + Z_1 + 3\,Z_2)\;I \tag{48}$$

Therefore

$$e_2 = \frac{Z_2\,I\;(1 - a)}{r} = \frac{2\,E}{r}\,\frac{Z_2}{L}\,\frac{1}{[p + (3\,Z_2 + Z_1)\;L]}$$

$$= \frac{2\,E}{r}\,\frac{Z_2}{Z_1 + 3\,Z_2}\left[1 - \epsilon^{-\frac{Z_1+3Z_2}{L}t}\right] \tag{49}$$

The fraction a may be found by writing the drop in phase B,

$$-a\,Z_1 - p\,L\,a + (1 - a)\,Z_2 = 0$$

Therefore

$$a = \frac{Z_2}{Z_1 + Z_2 + p\,L} \tag{50}$$

In Fig. 130 the amplitude factors and time constants for a number of three-phase bank connections have been tabulated. If two phases are simultaneously struck by lightning the secondary voltage will be the same as when one phase is struck. If a delta winding is present in either the primary or secondary or both, and all three phases are struck by equal lightning waves, no voltage will appear in the secondary circuit (ignoring the electrostatic transient).

APPENDIX TO CHAPTER XV

In Equation (43) appear two operational expressions which may be evaluated as follows

$$\left(\frac{p + \beta}{p + \alpha}\right)^n = \frac{(p + \beta)^n}{p}\,\frac{p}{(p + \alpha)^n} = \frac{(p + \beta)^n}{p}\left(\frac{t^{n-1}\,\epsilon^{-\alpha t}}{\underline{|n - 1}}\right) \tag{1}$$

and expanding by the binominal theorem

$$= \left[\frac{\beta^n}{p} + \sum_{k=1}^{n} \frac{\underline{|n}\,\beta^{n-k}\,p^{k-1}}{\underline{|k}\;\underline{|n-k}}\right]\left[\frac{t^{n-1}\,\epsilon^{-\alpha t}}{\underline{|n-1}}\right] \tag{2}$$

By Leibnitz's theorem, where $k \geq 1$

$$p^{k-1}\left(\frac{t^{n-1}\,\epsilon^{-\alpha t}}{\underline{|n-1}}\right) = \epsilon^{-\alpha t} \sum_{r=1}^{k} \frac{\underline{|k-1}\,(-\alpha)^{k-r}\,t^{n-r}}{\underline{|r-1}\;\underline{|k-r}\;\underline{|n-r}} \tag{3}$$

By integral calculus

$$\frac{\beta^n}{p}\left(\frac{t^{n-1}\,\epsilon^{-\alpha t}}{\underline{|n-1}}\right) = \frac{\beta^n}{\underline{|n-1}} \int_0^t t^{n-1}\,\epsilon^{-\alpha t}\,dt$$

$$= \frac{\beta^n}{\alpha^n} - \epsilon^{-\alpha t} \sum_{1}^{n} \frac{\beta^k}{\alpha^k} \frac{(\beta t)^{n-k}}{\underline{|n-k}} \tag{4}$$

Hence, substituting (3) and (4) in (2), there results

$$\left(\frac{p+\beta}{p+\alpha}\right)^n = \frac{\beta^n}{\alpha^n} - \epsilon^{-\alpha t}\left[\sum_{k=1}^{n} \frac{\beta^k}{\alpha^k} \frac{(\beta t)^{n-k}}{\underline{|n-k}} - \frac{\underline{|n}\;\beta^{n-k}}{\underline{|k}\;\underline{|n-k}} \sum_{r=0}^{k} \frac{\underline{|k-1}\;(-\alpha)^{k-r}\,t^{n-r}}{\underline{|r-1}\;\underline{|k-r}\;\underline{|n-r}}\right] \tag{5}$$

The other expression appearing in (43) is

$$\left(\frac{p+\beta-\alpha}{p}\right)^n = \sum_{0}^{n} \frac{\underline{|n}\,(\beta-\alpha)^{n-k}}{\underline{|k}\;\underline{|n-k}} \frac{1}{p^{n-k}}$$

$$= \sum_{0}^{n} \frac{\underline{|n}\,(\beta-\alpha)^{n-k}\,t^{n-k}}{\underline{|k}\;\underline{|n-k}\;\underline{|n-k}} \tag{6}$$

SUMMARY OF CHAPTER XV

An approximate equivalent circuit (Fig. 122*A*) for terminal transients has been set up by synthesis, and its elements identified quantitatively through the agency of the general solutions given in Chapter XII. This circuit, though too complex for actual computation purposes, nevertheless reduces to very simple equivalent circuits for the individual calculation of line reaction, neutral transient, and secondary terminal transients.

The reaction of a transformer at its primary line terminal is essentially that of a capacitance and inductance in parallel, the former being the effective terminal capacitance of the winding, and the latter its leakage inductance. The line terminal transient may therefore be divided into three principal stages: (1) the initial stage during which the capacitance elements are being charged, (2) an intermediate stage during which the capacitance elements are fully charged but the inductance has not yet allowed an appreciable increase in the current flowing through it, and (3) a final stage during which an influential current is flowing through the inductance. Thus, depending upon the time interval, a transformer may be said to act like a capacitance, an open circuit, an inductance and capacitance in parallel, and an inductance. If there is a choke coil or reactor in series with the transformer, the transformer may act essentially as a capacitance throughout the transient, by virtue of a high-frequency oscillation between that capacitance and the inductance of the series reactor or choke coil.

The equivalent circuit for calculating neutral transients consists of the leakage inductance of the transformer in series with a capacitance of such value that the frequency of oscillation of this L–C circuit is that of an isolated neutral transformer, and connected in shunt with this capacitance is the neutral impedance. This equivalent circuit can be easily solved for practical neutral impedances with constant parameters, but if the impedance contains some element such as Thyrite, then the neutral voltage is solved by a step-by-step process.

The equivalent circuit for calculating the transfer of waves through the transformer to the external secondary circuit consists simply of two mutually coupled inductances (assuming the neutral grounded and a closed secondary circuit, and ignoring the electrostatic component which is over with in a fraction of a microsecond). This becomes a "T" circuit when expressed on a 1 : 1 turn ratio basis; and since the staff thereof exercises but little influence on the character of the transient, for most practical purposes the circuit reduces to a series inductance equal to the leakage inductance between primary and secondary. Herefrom the transfer of waves, including repeated reflections, may be readily calculated. The concept is easily extended to include three-phase banks of transformers subjected to impulses on one, two, or three lines simultaneously. The further extension of the concept to multiple-winding transformers, although not carried out in the text, is obvious.

CHAPTER XVI

SUPPRESSION OF INTERNAL OSCILLATIONS BY ELECTROSTATIC SHIELDING

It is clear from the previous chapters that the internal transient oscillations caused by the impact of a traveling wave are responsible for two detrimental effects:

a. The envelope of oscillations everywhere exceeds the line of equilibrium corresponding to steady-state conditions, and therefore the major insulation from winding to ground can not be graded even approximately proportional to the steady-state voltage distribution. In the case of a grounded neutral transformer the internal voltage may exceed the line voltage by as much as 40 per cent, and in the case of an isolated neutral the excess may be greater than 125 per cent. Even these figures may be doubled if the applied wave is oscillatory and nearly in resonance with the fundamental natural period of the transformer.

b. The voltage gradients along the winding, the turn-to-turn and coil-to-coil stresses, are from 10 to 30 or more times the normal gradient corresponding to a uniform voltage distribution.

It is obvious, then, that considerable practical advantages would ensue if the harmonic oscillations could be suppressed.

The fundamental principle on which the constitutional remedy is based was first postulated by James Murry Weed.* He showed that a *necessary* condition for the absence of transient oscillations is:

$$\textit{initial distribution} = \textit{final distribution} \tag{1}$$

In practical cases this is also a *sufficient* condition.

Referring to Equation (39) of Chapter XIII it is seen that the amplitudes of oscillation vanish if

$$\alpha^2 - \beta^2 = 0 \tag{2}$$

* "Prevention of Transient Voltage in Windings," *A.I.E.E. Trans.*, Vol. 41, 1922.

But by (18) and (29) of Chapter XIII this is the condition for coincident initial and final distributions, as stated above. Rewriting α^2 and β^2 in terms of their definitions

$$\frac{C}{K} = \frac{rG}{1 + rg} \tag{3}$$

which shows that, theoretically, five circuit constants are available for control of the identity.

In a normal transformer, r and G are so small that $\beta \cong 0$, and any attempt to increase them artificially within the duration of the applied impulse must be discontinued at the cessation thereof, for otherwise they will cause excessive normal-frequency losses and seriously interfere with the functioning of the transformer. Of course it is a comparatively simple thing to substitute for g a Thyrite resistor which is effective at abnormal voltages, but does not influence the normal operating characteristics. However, a shunt resistor employed in this fashion destroys the oscillations by introducing excessive decrement factors, and not by insuring the identity called for by (1). Its utility is therefore in control of the envelope of oscillations, and it does not greatly reduce the gradients.

A much more profitable effort has been made with the capacitances C and K. If $\beta \cong 0$, then control of the transient requires

$$\alpha^2 = \frac{C}{K} \to 0 \tag{4}$$

and this condition may be realized either by decreasing C or increasing K. But C, the capacitance from the winding to ground, can not be reduced sufficiently to be of any advantage. Weed has pointed out, however, that it is not necessary to delete C, for its effects can be nullified by the addition of auxiliary electrostatic shields. At the present time there are installed several million kilovolt-amperes of transformers and auto-transformers employing electrostatic shielding. A transformer so protected is called a *non-resonating* transformer,* because it can not resonate with an applied wave of any shape, and the transient distribution is always constrained to be a straight line connecting the line and neutral voltages. In such a transformer it is obviously permissible to grade the major insulation in accordance with a linear distribution, thus realizing all the advantages incident thereto. The remainder of this chapter is devoted to an analysis of the different methods of shielding indicated in Fig. 131.

* "Effect of Transient Voltages on Power Transformer Design. I, II, III, IV," by K. K. Palueff, *A.I.E.E.Trans.*, Vols. 48, 49, 50, 51.

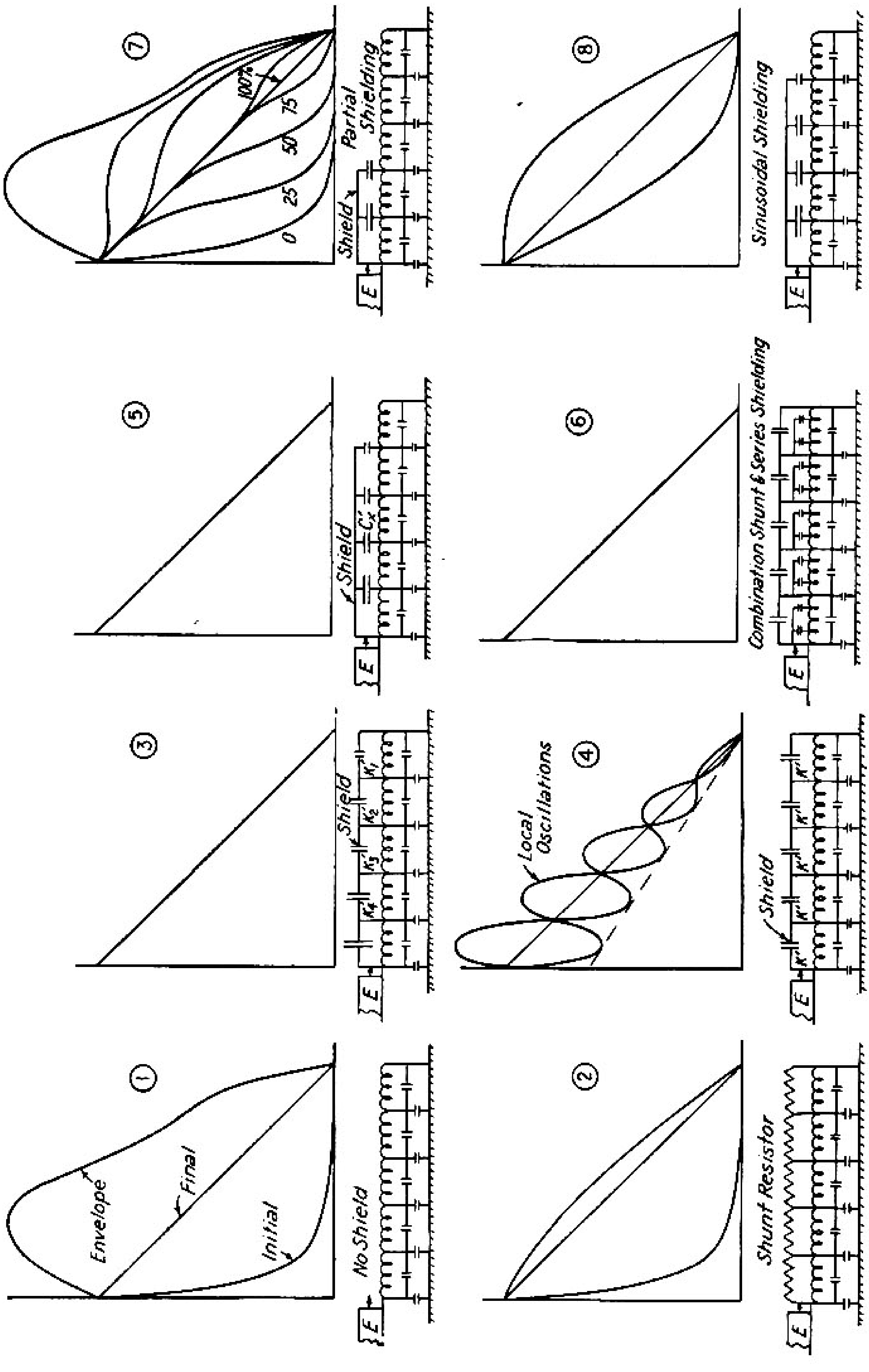


FIG. 131.—Methods of Shielding

Circuit 1 is an ordinary, or unshielded, grounded-neutral transformer. Since the initial (lowest curve) and final (straight line) distributions are different, a transient oscillation will take place, and these oscillations overshoot the final distribution so that abnormal voltages and gradients occur all along the stack.

Circuit 2 shows a tied-in shunt resistor, of such value that the oscillations are practically damped out and the envelope of oscillations definitely limited. (See Fig. 50 and Chapter V.) But the initial gradient at the line end is unchanged; and during the transient, steep gradients—but not so steep as in Circuit 1—occur progressively along the winding. For this reason a tied-in shunt resistor is not an ideal remedy.

Circuit 3 represents a tapered capacitance in shunt with the winding and assumed to be tied-in therewith at infinitesimal intervals so that no local oscillations between tied-in points are permitted. Suppose that the winding has been divided into n equal sections, and let the junctions be numbered $1, 2, \ldots, n+1$ from the neutral, and call the auxiliary capacitances $K'_1, K'_2, \ldots, K'_n$. Then for a linear distribution of voltage the conditions at junction x are

$$e = E/n \qquad = \text{volts per section.}$$

$$e_x = xE/n \qquad = \text{voltage to ground.}$$

$$E - e_x = \left(1 - \frac{x}{n}\right) E \qquad = \text{voltage to line terminal.}$$

By Kirchhoff's law, if C is the total capacitance of the winding to ground

$$\frac{C}{n} e_x = \frac{C}{n} ex = (nK + K_x') e - (nK + K'_{x-1}) e \tag{5}$$

Therefore

$$\begin{aligned} n K'_x &= xC + n K'_{x-1} \\ &= xC + (x-1) C + n K'_{x-2} \\ &= xC + (x-1) C + (x-2) C + n K'_{x-3} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= [x + (x-1) + (x-2) + \ldots + 1] C + 0 \\ &= \frac{x(x-1)}{2} C \end{aligned} \tag{6}$$

The total energy storage of the shield as $n \to \infty$ therefore is

$$W = \frac{1}{2}\int_0^1 K'_x\, e^2\, dx = \frac{1}{2}\int_0^1 \frac{x\,(x-1)}{2}\, C\, E^2\, dx$$

$$= \frac{5}{24}\, C\, E^2 \tag{7}$$

If, however, the shielding capacitances are tied-in at finite rather than infinitesimal intervals, this method of shielding gives rise to local oscillations of the same nature as discussed in the next case.

Circuit 4 employs a number of very large auxiliary capacitances K', and in effect simply increases the net series capacitance $K + K'$ of the transformer until $C/(K + K') \to 0$. These large capacitances may take the form of standard oil-filled capacitor units, such as used for power factor correction, and the number of tied-in points then would necessarily be limited. In that event the only points maintained on the linear distribution line at the initial instant would be these tied-in points, and intermediate points would be on a distorted initial distribution curve and therefore give rise to *local oscillations*. Let the fraction of the winding bridged by one capacitor unit be σ, distance from the end of the section as fraction of the length of section be y, and $\alpha = \sqrt{C/K}$ be the characteristic constant for the entire winding. Then at the nth section, counting from the neutral, the end voltages are

$$E_{n-1} = \sigma\, E\,(n-1)$$

$$E_n = \sigma\, E\, n$$

and by the principle of superposition the initial distribution over the section due to these two voltages is

$$e = \frac{\sigma\, E}{\sinh \alpha\, \sigma}\,[n \sinh \sigma\, \alpha\, y + (n-1) \sinh \sigma\, \alpha\,(1-y)] \tag{8}$$

Thus the smaller α or the greater the number of sections, the better the distribution, and the distortion decreases as the neutral is approached, that is, as $n \to 1$, as is seen from Fig. 131–4. If the lowest natural frequency of a short section of a winding is very high—which may be the case, since the frequency increases almost as the square of the space harmonic—then a few microseconds' depression of the applied wave front would eliminate these very high-frequency local oscillations. The effective capacitance of a transformer shielded in this fashion should be made sufficiently high to accomplish the desired depression of wave front for all waves arriving over the surge impedance of a transmission line. In Fig. 132 the reduction factor corresponding to the effective capacitance and the natural frequency of oscillations has been plotted. For example, suppose that a transformer ($\alpha = 10$) having a fundamental natural frequency of 5000 ~ is to be shielded by five capacitor units of 0.10 microfarad each. By Fig. 107 the fifth harmonic will oscillate at a frequency of

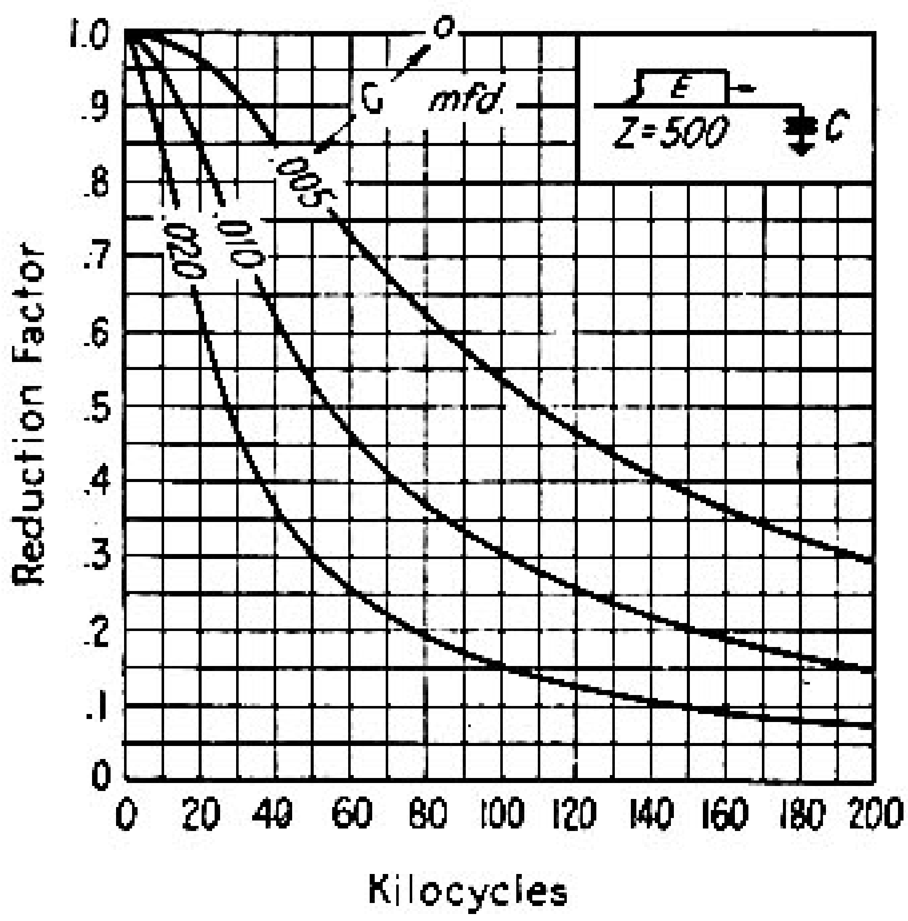


FIG. 132.—Reduction Factor for Oscillations

$$13 \times 5000 = 65{,}000 \sim$$

and the effective capacitance of the transformer will be approximately

$$\frac{0.10}{5} = 0.02$$

By Fig. 132 the reduction factor is 0.24, which means that the amplitude of the local oscillations will be less than a quarter of those shown in Fig. 131–4.

This method of shielding is also applicable to delta-connected and isolated neutral transformers.

Circuit 5 represents the conventional method of shielding as employed on non-resonating transformers. Assume that the capacitances of the winding are not uniformly distributed (as would actually be the case in a transformer with graded insulation), and let

x_1, x_2, x_3 = fraction of winding from the neutral for any adjacent coils 1, 2, 3 respectively.

C_1, C_2, C_3 = capacitances from coils to ground.

K_{12}, K_{23} = capacitances between adjacent coils.

e_1, e_2, e_3 = voltages above ground of adjacent coils.

C_1', C_2', C_3' = capacitance from shield to coils.

Now the currents entering and leaving Coil 2 must balance, so that

$$K_{12}(e_1 - e_2) + C_2'(E - e_2) = K_{23}(e_2 - e_3) + C_2 e_2 \tag{9}$$

$$C_2' = \frac{C_2 e_2 + K_{23}(e_2 - e_3) - K_{12}(e_1 - e_2)}{(E - e_2)} \tag{10}$$

from which the necessary shield capacitance for any distribution of capacitances and voltage may be determined. If the distribution is linear, then

$$\left.\begin{aligned} e_1 &= x_1 E \\ e_2 &= x_2 E \\ e_3 &= x_3 E \end{aligned}\right\}$$

and (10) reduces to

$$\begin{aligned} C_2' &= \frac{x_2}{1 - x_2} C_2 + \frac{x_2 - x_1}{1 - x_2} K_{12} + \frac{x_2 - x_3}{1 - x_2} K_{23} \\ &\cong \frac{x_2}{1 - x_2} C_2 \end{aligned} \tag{11}$$

This same result follows from differential equation (17) upon substituting $e = x\,E$. The energy storage of the shield is

$$W = \frac{1}{2}\int_0^1 C_x' \,(E - e_x)^2\, dx = \frac{C\,E^2}{12} \tag{12}$$

The advantages of this method of shielding are that it eliminates local oscillations, and it can be designed as an integral part of the coil and insulation structure of the transformer.

Circuit 6 is a combination of Circuits 4 and 5, in which the local oscillations have been eliminated by the addition of conventional shields for each section. These auxiliary shields do not require to be insulated from the winding for more than the voltage between tied-in points. In view of the discussion covering Circuit 4 it is doubtful if this combination is justifiable, except possibly on the line end sections.

Circuit 7 represents *partial* or incomplete conventional shielding, in which the shielding has been carried only part way down the stack. The distributions can be calculated (assuming a uniform winding) as follows

$$i_k = K \frac{\partial^2 e}{\partial x\, \partial t} = \text{current in series capacitance.} \tag{13}$$

$$i_c = C \frac{\partial e}{\partial t} = \text{current to ground.} \tag{14}$$

$$i' = C'(x) \frac{\partial}{\partial t}(E - e) = \text{current from shield.} \tag{15}$$

$$i_c = i' + \frac{\partial i_k}{\partial x} \tag{16}$$

Therefore, after canceling the operator $(\partial/\partial t)$

$$\frac{d^2 e}{dx^2} - \frac{C + C'(x)}{K} e = - \frac{C'(x)}{K} E \tag{17}$$

which is the differential equation that must be satisfied for any distribution of shielding $C'(x)$. If the shielding is discontinued at point $x = g$ it is convenient to divide the circuit in two parts as follows:

$$\left.\begin{aligned} &\text{Region 1 from } x = g \text{ to } x = 1 \\ &\text{Region 2 from } x = 0 \text{ to } x = g \end{aligned}\right\} \tag{18}$$

and the corresponding differential equations and boundary conditions are

$$\frac{d^2 e_1}{dx^2} - \frac{C + C'(x)}{K} e = - \frac{C'(x)}{K} E \tag{19}$$

$$\frac{d^2 e_2}{dx^2} - \frac{C}{K} e = 0 \tag{20}$$

$$\left.\begin{aligned} e_1 &= E \quad \text{at } x = 1 \\ e_2 &= 0 \quad \text{at } x = 0 \\ e_1 &= e_2 \quad \text{at } x = g \\ \frac{de_1}{dx} &= \frac{de_2}{dx} \quad \text{at } x = g \end{aligned}\right\} \tag{21}$$

Now for linear shielding

$$C'(x) = \frac{x}{1 - x} C \tag{22}$$

and (19) then becomes $(\alpha^2 = C/K)$,

$$\frac{d^2 e_1}{dx^2} - \frac{\alpha^2 e_1}{1 - x} = - \frac{\alpha^2 x}{1 - x} E \tag{23}$$

Let

$$y = \frac{e_1}{E} - x$$

which reduces (23) to Riccati's equation

$$\frac{d^2 y}{dx^2} - \frac{\alpha^2 y}{1 - x} = 0 \tag{24}$$

Changing the independent variable to

$$x = 1 + t^2/4\,\alpha^2 \tag{25}$$

there is

$$\frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt} + y = 0 \tag{26}$$

and now changing the dependent variable to

$$y = vt \tag{27}$$

there results Bessel's equation of the first order

$$t^2\frac{d^2 v}{dt^2} + t\frac{dv}{dt} + v\,(t^2 - 1) = 0 \tag{28}$$

the solution to which is

$$v = A\ J_1(t) + B\ K_1(t) \tag{29}$$

where A, B are integration constants, and J_1 and K_1 are Bessel functions of the first order and first and second kind respectively. Changing the variables in (29) back to the originals

$$e_1 = E\,[x + (2\,\alpha\sqrt{x-1})\ A\ \ J_1(2\ \alpha\sqrt{x-1})$$
$$+ B\ (2\,\alpha\sqrt{x-1})\ K_1\ (2\,\alpha\sqrt{x-1})] \tag{30}$$

At $x = 1$, $e_1 = E$. But $J_1\ (0) = 0$ and $(0)\ K_1\ (0) = 1$. Therefore $B = 0$ and

$$e_1 = E\,[x + A\ 2\,\alpha\sqrt{x-1}\ \ J_1(2\ \alpha\sqrt{x-1})] \tag{31}$$

For large values of the argument the Bessel functions may be expanded as an asymptotic series, and since the argument is purely imaginary

$$\left.\begin{aligned} 2\,\alpha\sqrt{x-1}\ &= j\,2\,\alpha\sqrt{1-x} = j\,z \\ J_n(j\,z)\ &= j^n\ I_n(z) \\ (j\,z)\ J_1(j\,z)\ &= -\ z\ I_1(z) \end{aligned}\right\} \tag{32}$$

A suitable asymptotic expansion is *

$$\mathrm{I}_n(z)$$
$$= \frac{\epsilon^z}{\sqrt{2\,\pi\,z}}\left[1 + \sum_1^\infty \frac{(-)^r\,(4\,n^2-1^2)\,(4\,n^2-3^2)\ldots(4\,n^2-\overline{2\,r-1}^2)}{\underline{|r}\ 2^{3r}\,z^r}\right] \tag{33}$$

also

$$\frac{d}{dz}\,[z^n\ I_n(z)] = z^n\ I_{n-1}\ (z) \tag{34}$$

* "Modern Analysis," by Whittaker and Watson, Cambridge University Press.

The solution to (20) is

$$e_2 = A_2 \sinh \alpha x + B_2 \cosh \alpha x \tag{35}$$

and by the boundary conditions $B_2 = 0$, and at $x = g$

$$E\,[g - A\ 2\,\alpha\sqrt{1-g}\ I_1(2\,\alpha\sqrt{1-g})] = A_2 \sinh \alpha g \tag{36}$$

$$E\,[1 + A\ 2\ \alpha^2\ I_0(2\,\alpha\sqrt{1-g})] = \alpha\,A_2 \cosh \alpha g \tag{37}$$

Herefrom

$$A = \frac{(\alpha g - \tanh \alpha g)}{2\,\alpha^2\ I_0(2\,\alpha\sqrt{1-g})\ \tanh \alpha g + \alpha(2\,\alpha\sqrt{1-g})\ I_1(2\,\alpha\sqrt{1-g})} \tag{38}$$

$$A_2 = \frac{[2\,\alpha^2 g\ I_0(2\,\alpha\sqrt{1-g}) + (2\,\alpha\sqrt{1-g}) I_1(2\,\alpha\sqrt{1-g})]\ \mathrm{sech}\ \alpha g}{2\,\alpha^2\ I_0(2\,\alpha\sqrt{1-g})\ \tanh \alpha g + \alpha(2\,\alpha\sqrt{1-g})\ I_1(2\,\alpha\sqrt{1-g})} \tag{39}$$

$$e_1 = E\,[x - A\,(2\,\alpha\sqrt{1-x})\ \ I_1(2\,\alpha\sqrt{1-x})] \tag{40}$$

$$e_2 = A_2 \frac{\sinh \alpha x}{\sinh \alpha g} \tag{41}$$

As an example take $g = 0.5$, $\alpha = 10$. Then

$$2\,\alpha\sqrt{1-g} = 14.14, \quad \alpha g = 5, \quad \tanh \alpha g = 1$$

$$\left.\begin{aligned} I_0\ (14.14) &= \frac{\epsilon^{14.14}}{\sqrt{2\,\pi\ 14.14}}\left[1 + \frac{1}{8\ (14.14)} + \frac{9}{128\ (14.14)^2} + \ldots\right] \\ I_1\ (14.14) &= \frac{\epsilon^{14.14}}{\sqrt{2\,\pi\ 14.14}}\left[1 + \frac{3}{8\ (14.14)} - \frac{15}{128\ (14.14)^2} + \ldots\right] \end{aligned}\right\} \tag{42}$$

$$\begin{aligned} e_1 &= E\left[x - \frac{\sqrt{1-x}}{4.27}\ \frac{I_1\ (20\sqrt{1-x})}{I_1\ (14.14)}\right] \\ &\cong x\,E - \frac{E\sqrt[4]{1-x}}{5.08}\ \epsilon^{(20\sqrt{1-x} - 14.14)} \end{aligned} \tag{43}$$

$$e_2 = 0.335\ E \frac{\sinh 10\,x}{\sinh 5} \tag{44}$$

Circuit 8 shows another possibility in the direction of partial shielding in which the shield is the full length of the winding but of reduced capacitance, so that the compensation is not complete.

Static Plates.—A static plate at the end of the coil stack was first used as a mechanical support to the stack and as a convenient means of providing insulation to ground. It was later discovered to be of considerable benefit in equalizing the distribution over the turns of the first few sections. In long stacks with narrow coils a static plate does not exercise any appreciable influence on the characteristics of the transient oscillations. In short stacks with wide coils a static plate is effective in reducing the fundamental and first few harmonics, but it increases the harmonics contained by the individual sections and therefore exaggerates the local oscillations.

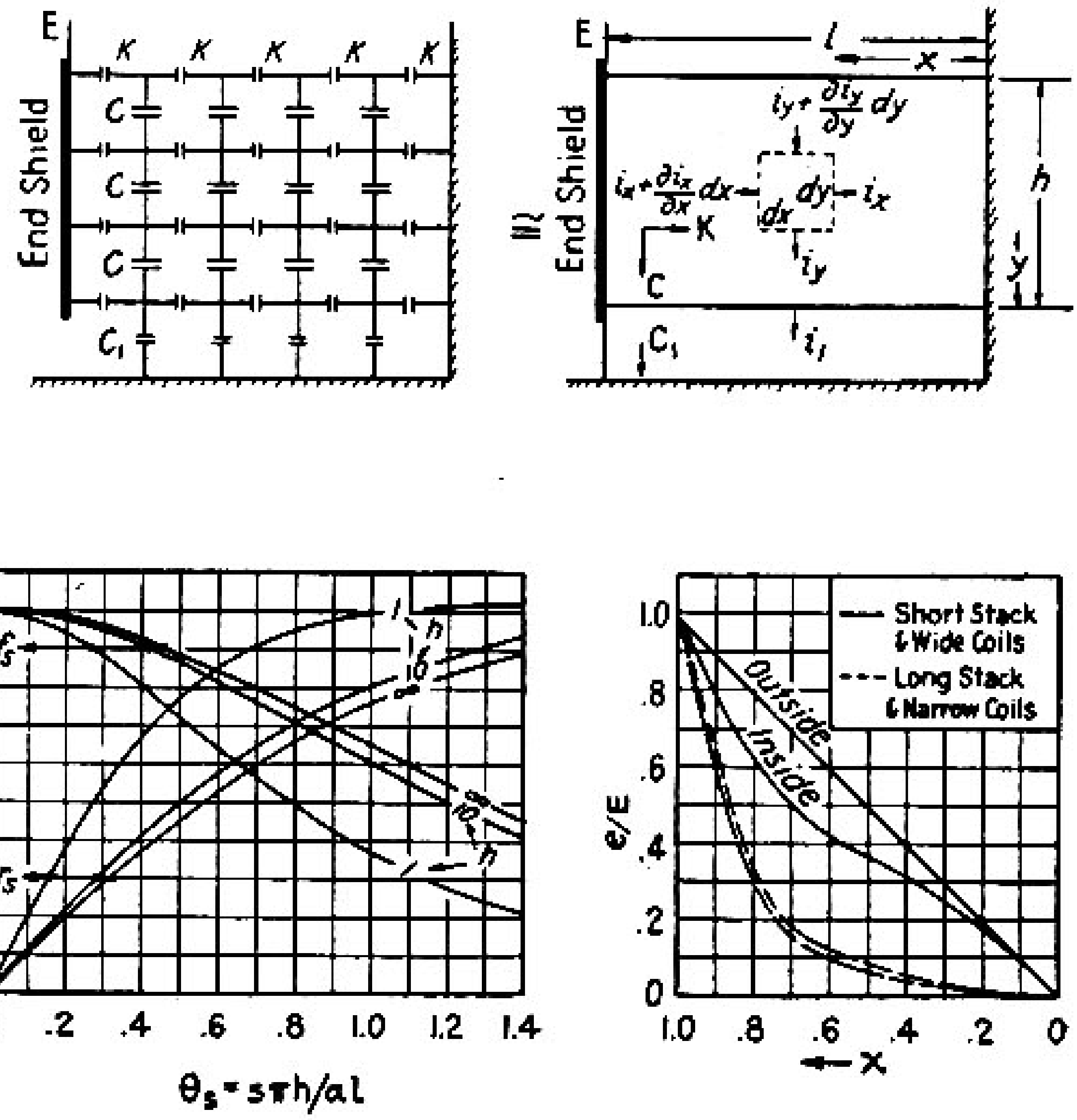


FIG. 133.—Effect of Static Plate

Fig. 133 shows the capacitance network of a coil stack equipped with a static plate, in which K and C are respectively the axial and radial turn-to-turn capacitances, and C_1 is the capacitance to ground. This network is replaced by the solid dielectric of Fig. 133, assumed to have the crystalline property of different specific capacitances in two orthogonal directions of flow. For the element $dx\,dy$ the divergence (excess of outward to inward current) is zero, therefore

$$\nabla \cdot i = \frac{\partial i_y}{\partial y} + \frac{\partial i_x}{\partial x} = 0 \tag{45}$$

and the two current components are

$$i_y = C \frac{\partial e}{\partial y} \tag{46}$$

$$i_x = K \frac{\partial e}{\partial x} \tag{47}$$

so that (45) becomes

$$C \frac{\partial^2 e}{\partial y^2} + K \frac{\partial^2 e}{\partial x^2} = 0 \tag{48}$$

subject to the boundary conditions

$$\left.\begin{aligned} e &= 0 \text{ at } x = 0 \\ e &= E \text{ at } x = l \end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned} \frac{\partial i_x}{\partial x} + i_y &= i_1 \text{ at } y = 0 \\ \frac{\partial i_x}{\partial x} - i_y &= 0 \text{ at } y = h \end{aligned}\right\} \tag{50}$$

Substituting (46) and (47) in (50) and writing

$$\left.\begin{aligned} \alpha^2 &= C/K \\ \beta^2 &= C_1/K \end{aligned}\right\} \tag{51}$$

the differential equation and its boundary conditions become

$$\frac{\partial^2 e}{\partial x^2} + \alpha^2 \frac{\partial^2 e}{\partial y^2} = 0 \tag{52}$$

$$e = 0 \quad \text{at } x = 0 \tag{53}$$

$$e = E \quad \text{at } x = l \tag{54}$$

$$\frac{\partial^2 e}{\partial x^2} + \alpha^2 \frac{\partial e}{\partial y} = \beta^2 e \text{ at } y = 0 \tag{55}$$

$$\frac{\partial^2 e}{\partial x^2} - \alpha^2 \frac{\partial e}{\partial y} = 0 \quad \text{at } y = h \tag{56}$$

Assume as a tentative solution

$$e = A_0 + B_0\, x + C_0\, y + D_0\, xy + \sum (A \sin ax + B \cos ax)$$
$$(C \sinh by + D \cosh by) \tag{57}$$

which substituted in (52) yields

$$b = \frac{a}{\alpha}$$

According to (53)

$$A_0 = C_0 = B = 0$$

According to (54)

$$E = B_0\, l + y\, D_0\, l + \sum A \sin al\, (C \sinh by + D \cosh by)$$

hence

$$B_0 = \frac{E}{l}, \quad D_0 = 0, \quad a = \frac{s\,\pi}{l}, \quad A = 1$$

and the solution takes the form

$$e = \frac{x}{l} E + \sum \left(C \sinh \frac{s\,\pi\, y}{\alpha\, l} + D \cosh \frac{s\,\pi\, y}{\alpha\, l} \right) \sin \frac{s\,\pi\, x}{l} \tag{58}$$

Substituting in (56)

$$-\frac{C}{D} = \left[\frac{h \tanh \theta_s + \theta_s}{\theta_s \tanh \theta_s + h} \right] = r_s \tag{59}$$

where

$$\theta_s = \frac{s\,\pi\, h}{\alpha\, l} \tag{60}$$

Now expressing (x/l) as a half-range *sine* series

$$\frac{x}{l} E = \sum_1^\infty \frac{-2\,E}{s\,\pi} (-1)^s \sin \frac{s\,\pi\, x}{l} \tag{61}$$

substituting (58) in (55) and making use of (61) there results

$$C \frac{\alpha\, s\,\pi}{l} - D \left(\frac{s^2\,\pi^2}{l^2} + \beta^2 \right) = -\, (-1)^s \frac{2\,E\,\beta^2}{s\,\pi} \tag{62}$$

Solving (59) and (62) as simultaneous equations

$$-\,C = r_s\, D = (-1)^s \frac{2\,E}{s\,\pi} \left[\frac{\beta^2\, r_s}{\beta^2 + \dfrac{s^2\,\pi^2}{l^2} + r_s \dfrac{\alpha\, s\,\pi}{l}} \right] \tag{63}$$

and hereby the solution becomes

$$e = \frac{x}{l} E + \sum_1^\infty \frac{2\,E}{s\,\pi} (-1)^s \left[\frac{\beta^2}{\beta^2 + \dfrac{s^2\,\pi^2}{l^2} + r_s \dfrac{\alpha\, s\,\pi}{l}} \right]$$

$$\left(-\, r_s \sinh \frac{s\,\pi\, y}{\alpha\, l} + \cosh \frac{s\,\pi\, y}{\alpha\, l} \right) \sin \frac{s\,\pi\, x}{l} \tag{64}$$

In particular, the upper and lower boundaries of the axial distribution are of interest

$$\underset{y=0}{e} = \frac{x}{l} E + \sum_{1}^{\infty} \frac{2E}{s\pi} (-1)^s \left[\frac{\beta^2}{\beta^2 + \frac{s^2 \pi^2}{l^2} + r_s \frac{\alpha s \pi}{l}} \right] \sin \frac{s \pi x}{l} \tag{65}$$

$$\underset{y=h}{e} = \frac{x}{l} E + \sum_{1}^{\infty} \frac{2E}{s\pi} (-1)^s \left[\frac{\beta^2 f_s}{\beta^2 + \frac{s^2 \pi^2}{l^2} + r_s \frac{\alpha s \pi}{l}} \right] \sin \frac{s \pi x}{l} \tag{66}$$

where

$$f_s = (-r_s \sinh \theta_s + \cosh \theta_s) \tag{67}$$

The functions r_s and f_s from (59) and (67) respectively have been plotted in Fig. 133 as functions of θ_s.

The foregoing analysis of the effect of a static plate assumed ground adjacent to the coil stack only on the neutral end and opposite the inner cross-overs. It was further tacitly assumed, in the interests of mathematical simplicity, that the distance of the last coil section (pancake) from the ground plate was the same as the interval between adjacent sections. Nevertheless, the solutions obtained may be easily interpreted for the case of ground on both sides of the stack, and the last coil any distance from the ground plate. For suppose that ground is equal distance either side of the coil, as is the case in a normal shell-type design. Then obviously, by symmetry, there will be no radial component of dielectric flux at half the radial depth of the coil, and therefore each half of the coil is independent, and its distribution may be computed as in Fig. 134. Thus, with iron on both sides of the coil, the inside and outside distributions coincide, and it is the midpoint of the section which is at highest potential. The maximum difference of potential between any two points of the section is between the midpoint and either the inside or outside cross-over, and this difference is the same as for a coil of half the radial build and with iron on one side only.

In order to account approximately for a variable distance from the last section to the ground plate, it is only necessary to appreciate the fact that the x and y distances in the "flow sheet" of Fig. 133 are in reality axial and radial thickness of *insulation*, that is, do not include the width and thickness of the copper conductors. On this basis, if the capacitance of the last section to the ground plate is K_1 then its *equivalent* distance from the ground plate in the "flow sheet" diagram may be taken as

$$\frac{x_0}{l} = \frac{K}{K_1}$$

where K is the axial capacitance from static plate to ground plate. This, of course, is a rough approximation, but will serve to show the effect of increasing the axial distance to ground. It is thus evident that an increase in the axial distance from the last section to the ground plate is equivalent to a longer coil stack, and if (x_0/l) is about equal to 1/2, the greatest potential difference will occur in the last section. If the last section is actually grounded, then this potential difference becomes very excessive indeed. Test data, and solutions from both dielectric field plots and a d-c. calculating board, have been given by K. K. Palueff,* and they show how the irregularities become greater for the wider and shorter coil stacks.

It may be of interest to point out that the effect of a static plate has been investigated by at least five different methods:

1. Tests on actual coil stacks, both with sphere gaps and the cathode-ray oscillograph.
2. Solution of the capacitance network by successive trials.
3. Solution of the capacitance network by substituting the corresponding resistance network on the d-c. calculating board.
4. Dielectric flux plots.
5. Analytic solution given in this book.

Effective Capacitance of Shielded Windings.—It was shown in Chapter XIII that the effective capacitance of an ordinary transformer without a shield is

$$\left.\begin{aligned} C_{\text{eff}} &= \sqrt{C\,K}\tanh\alpha \quad \text{for isolated neutral} \\ &= \sqrt{C\,K}\coth\alpha \quad \text{for grounded neutral} \end{aligned}\right\} \tag{68}$$

and since α is of the order of five or more

$$C_{\text{eff}} \cong \sqrt{C\,K} = \frac{C}{\alpha} \tag{69}$$

If a transformer is shielded by the method of Fig. 131–4, the effective capacitance is

$$C'_{\text{eff}} = \sqrt{C\,(K + K')} \tag{70}$$

and

$$\frac{C'_{\text{eff}}}{C_{\text{eff}}} = \sqrt{1 + \frac{K'}{K}} \tag{71}$$

* See References 5, 7, 11, 17 in the Bibliography and the discussions of these papers in the *A.I.E.E. Trans.*

If a transformer is shielded by the method of Fig. 131-5 the effective capacitance (from shield through winding to ground) is

$$C''_{\text{eff}} = K + \int_0^1 x\, C\, dx = K + \frac{C}{2} \cong \frac{C}{2} \tag{73}$$

and

$$\frac{C''_{\text{eff}}}{C_{\text{eff}}} = \frac{K + C/2}{C/\alpha} = \frac{\alpha}{2} + \frac{1}{\alpha} \cong \frac{\alpha}{2} \tag{74}$$

However, the capacitance from the shield to the tank practically doubles this value. Therefore, the shield increases the effective capacitance α times, which means that a non-resonating transformer may have from 10 to 20 times or more the capacitance of an ordinary transformer of the same rating.

SUMMARY OF CHAPTER XVI

If the *initial distribution* is made equal to the *final distribution* for an infinite rectangular applied wave, then no oscillations can occur. This may be brought about by means of *electrostatic shields*, and, as illustrated in Fig. 131, these shields may be applied in several different ways. In general, there are two classes of shields:

1. Those which furnish the charging currents of the capacitance C from winding to ground, thereby relieving the series capacitance K of these currents, and so permitting it to establish a linear distribution of voltage.
2. Those which reinforce the capacitance K to such an extent that the ground capacitance C can exert no appreciable influence on the distribution, and it therefore remains substantially linear.

The advantages and disadvantages of the different methods of shielding were discussed briefly in the text; and the formulas applying to each case were derived. In the case of partial shielding, Fig. 131-7, a great deal of space was devoted to mathematical derivations, which may inadvertently have given an entirely erroneous idea of the relative importance of this practice as compared with the other methods of shielding which were described.

The chief advantage of the *static plate* is in helping to equalize the voltage distribution on the turns of the first few sections. It is ineffective in exercising any appreciable influence on the characteristics of the oscillations in long, narrow coil stacks, but in short, wide coil stacks it improves the *average* distribution at the expense of exaggerated local oscillations.

The effective capacitance of a non-resonating transformer may be from 10 to 20 or more times that of an ordinary transformer of the same rating.

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APPENDIX

The following table of operational calculus and other useful formulas has been compiled, with the kind permission of the authors and publishers, from similar tables given in:

"Heaviside's Operational Calculus as Applied to Engineering and Physics," by E. J. Berg, McGraw-Hill Book Co.

"Operational Circuit Analysis," by Vannevar Bush, John Wiley & Sons.

A knowledge of operational calculus to the extent covered in these fine books will be found well-nigh indispensable in the study of traveling waves.

HYPERBOLIC AND ALLIED FUNCTIONS

(1) $\epsilon^{x} = 1 + x + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}} + \ldots = \cosh x + \sinh x$

(2) $\epsilon^{jx} = 1 + jx - \dfrac{x^2}{\underline{|2}} - j\dfrac{x^3}{\underline{|3}} + \ldots = \cos x + j \sin x$

(3) $\sinh x = \dfrac{\epsilon^{x} - \epsilon^{-x}}{2} = x + \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} + \ldots = x \displaystyle\prod_{1}^{\infty}\left(1 + \frac{x^2}{\pi^2 s^2}\right)$

(4) $\cosh x = \dfrac{\epsilon^{x} + \epsilon^{-x}}{2} = 1 + \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} + \ldots = \displaystyle\prod_{1}^{\infty}\left(1 + \frac{4x^2}{\pi^2 (2s - 1)^2}\right)$

(5) $\sin x = \dfrac{\epsilon^{jx} - \epsilon^{-jx}}{2j} = x - \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} - \ldots = x \displaystyle\prod_{1}^{\infty}\left(1 - \frac{x^2}{s^2 \pi^2}\right)$

(6) $\cos x = \dfrac{\epsilon^{jx} + \epsilon^{-jx}}{2} = 1 - \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} - \ldots = \displaystyle\prod_{1}^{\infty}\left(1 - \frac{4x^2}{\pi^2 (2s - 1)^2}\right)$

(7) $\sinh jx = j \sin x$

(8) $\cosh jx = \cos x$

(9) $\sin jx = j \sinh x$

(10) $\cos jx = \cosh x$

(11) $\sinh(-x) = -\sinh x$

(12) $\cosh(-x) = \cosh x$

(13) $\cosh^2 x - \sinh^2 x = 1$

(14) $\dfrac{d}{dx} \sinh a\,x = a \cosh a\,x$

(15) $\dfrac{d}{dx} \cosh a\,x = a \sinh a\,x$

(16) $\displaystyle\int \sinh a\,x\,dx = \frac{1}{a} \cosh a\,x$

(17) $\displaystyle\int \cosh a\,x\,dx = \frac{1}{a} \sinh a\,x$

(18) $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(19) $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(20) $\sinh (x \pm j\,y) = \sinh x \cos y \pm j \cosh x \sin y$

$$= \sqrt{\sinh^2 x + \sin^2 y}\;\underline{\lfloor \phi_1}$$

(21) $\cosh (x \pm j\,y) = \cosh x \cos y \pm j \sinh x \sin y$

$$= \sqrt{\cosh^2 x - \sin^2 y}\;\underline{\lfloor \phi_2}$$

(22) $\sin (x \pm j\,y) = \sin x \cosh y \pm j \cos x \sinh y$

$$= \sqrt{\sin^2 x + \sinh^2 y}\;\underline{\lfloor \phi_3}$$

(23) $\cos (x \pm j\,y) = \cos x \cosh y \mp j \sin x \sinh y$

$$= \sqrt{\cos^2 x + \sinh^2 y}\;\underline{\lfloor \phi_4}$$

where

$$\tan \phi_1 = \pm \coth x \,.\, \tan y$$

$$\tan \phi_2 = \pm \tanh x \,.\, \tan y$$

$$\tan \phi_3 = \pm \tanh y \,.\, \cot x$$

$$\tan \phi_4 = \mp \tanh y \,.\, \tan x$$

FACTORIALS

(24) $\underline{\lfloor x} = \Pi (x) = \Gamma (1 + x) = \lim\limits_{n \to \infty} \left[\dfrac{1 \cdot 2 \cdot 3 \cdot \;\ldots\; (n-1)}{(x+1) \ldots (x+n-1)}\, n^x \right]$

(25) $\underline{\lfloor (2\,n - 1)/2} = \sqrt{\pi}\, \dfrac{1 \cdot 3 \cdot 5 \cdot \;\ldots\; (2\,n-1)}{2 \cdot 2 \cdot 2 \cdot \;\ldots\; 2}$

(26) $\underline{\lfloor -(2\,n - 1)/2} = \sqrt{\pi}\, \dfrac{2 \cdot 2 \cdot 2 \cdot \;\ldots\; 2}{1 \cdot 3 \cdot 5 \cdot \;\ldots\; (2\,n-1)} \sin (2\,n - 1) \dfrac{\pi}{2}$

$$= \frac{\pi}{\underline{\lfloor (2\,n - 3)/2}} \sin (2\,n - 1) \frac{\pi}{2}$$

(27)

x	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	-1	0
$\lfloor\underline{x}$	$\sqrt{\pi}\,\frac{1}{2}$	$\sqrt{\pi}\,\frac{3}{4}$	$\sqrt{\pi}\,\frac{15}{8}$	$\sqrt{\pi}$	$-2\sqrt{\pi}$	$\frac{4}{3}\sqrt{\pi}$	∞	1

EXPANSIONS

Fourier series

$$(28)\quad f(x) = \frac{b_0}{2} + \sum\left(b_n \cos\frac{n\pi x}{c} + a_n \sin\frac{n\pi x}{c}\right)$$

where
$$b_n = \frac{1}{c}\int_{-c}^{c} f(y)\cdot\cos\frac{n\pi y}{c}\,dy$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(y)\cdot\sin\frac{n\pi y}{c}\,dy$$

Taylor's series

$$(29)\quad f(x+a) = f(x) + a f'(x) + \frac{a^2}{\lfloor\underline{2}}f''(x) + \ldots + \frac{a^n}{\lfloor\underline{n}}f^n(x+\theta a)$$

$$(30)\quad f(x+a,\ y+b,\ z+c,\ \ldots,\ w+n) = \epsilon^{\sigma} f(x, y, z, \ldots, w)$$

where
$$\sigma = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + \ldots + n\frac{\partial}{\partial w}\right)$$

Maclaurin's series

$$(31)\quad f(x) = f(o) + \frac{x}{1}f'(o) + \frac{x^2}{\lfloor\underline{2}}f''(o) + \ldots + \frac{x^{n-1}}{\lfloor\underline{n-1}}f^{n-1}(o) + \frac{x^n}{\lfloor\underline{n}}f''(\theta x)$$

Binomial theorem

$$(32)\quad (x+y)^n = \sum_0^n \frac{\lfloor\underline{n}\, x^{n-k}\, y^k}{\lfloor\underline{k}\ \lfloor\underline{n-k}} = x^n + n\,x^{n-1}\,y + \frac{n(n-1)\,x^{n-2}\,y^2}{\lfloor\underline{2}} + \ldots$$

Leibnitz's theorem

$$(33)\quad \frac{d^n(u\,v)}{d\,x^n} = \sum_0^n \frac{\lfloor\underline{n}\, D^{n-k}\,u\,D^k v}{\lfloor\underline{k}\ \lfloor\underline{n-k}}$$

$$(34)\quad \int \epsilon^{ax}\,X\,dx = \frac{\epsilon^{ax}}{a}\left[X - \frac{1}{a}\frac{d\,X}{d\,x} + \frac{1}{a^2}\frac{d^2\,X}{d\,x^2} - \frac{1}{a^3}\frac{d^3\,X}{d\,x^3} + \ldots\right]$$

OPERATIONAL CALCULUS

Expansion theorem

$$(35)\quad \frac{Y(p)}{Z(p)} = \frac{Y(o)}{Z(o)} + \sum_{p_1, p_2, \ldots} \frac{Y(p_k)\,\epsilon^{p_k t}}{p_k Z'(p_k)}$$

$$\frac{1}{H(p)} = \frac{1}{H(o)} + \sum_{p_1, p_2, \ldots} \frac{\epsilon^{p_k t}}{p_k H'(p_k)}$$

Duhamel's theorem

$$(36)\quad f(t) = E(o)\,\phi(t) + \int_0^t \phi(t-\tau)\frac{\partial}{\partial\tau}E(\tau)\,d\tau$$

$$= E(o)\,\phi(t) + \int_0^t \phi(\tau)\frac{\partial}{\partial t}E(t-\tau)\cdot d\tau$$

$$= E(t)\,\phi(o) + \int_0^t E(\tau)\frac{\partial}{\partial t}\phi(t-\tau)\,d\tau$$

$$= E(t)\,\phi(o) + \int_0^t E(t-\tau)\frac{\partial}{\partial\tau}\phi(\tau)\,d\tau$$

$$= \frac{d}{dt}\int_0^t E(\tau)\,\phi(t-\tau)\,d\tau$$

$$= \frac{d}{dt}\int_0^t E(t-\tau)\,\phi(\tau)\,d\tau$$

Shifting theorem

$$(37)\quad f(p)\left[\phi(t)\,\epsilon^{\alpha t}\,\mathbf{1}\right] = \epsilon^{\alpha t} f(p+\alpha)\cdot\left[\phi(t)\,\mathbf{1}\right]$$

$$(38)\quad p^m\,\frac{t^n}{\lfloor\underline{n}} = \frac{t^{n-m}}{\lfloor\underline{n-m}}$$

$$(39)\quad p^m\,\mathbf{1} = \frac{t^{-m}}{\Gamma(1-m)} \quad \text{except for } m \text{ a positive integer}$$

$$(40)\quad p^m\,t^n\,\mathbf{1} = \frac{\Gamma(n+1)\,t^{n-m}}{\Gamma(n-m+1)}$$

$$(41)\quad p^m f(t)\,\mathbf{1} = \sum_0^\infty a_n \frac{\Gamma(n+1)\,t^{n-m}}{\Gamma(n-m+1)} \quad \text{if } f(t) = \sum_0^\infty a_n t^n$$

$$(42)\quad p^{1/2} f(t)\,\mathbf{1} = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{f(\tau)\,d\tau}{\sqrt{t-\tau}}$$

$$(43)\quad p^{1/2}\,\mathbf{1} = \sqrt{\frac{2}{\pi}}\,\frac{1}{(2t)^{1/2}}$$

(44) $p^{3/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,\dfrac{-1}{(2t)^{3/2}}$

(45) $p^{5/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,\dfrac{1\cdot 3}{(2t)^{5/2}}$

(46) $p^{7/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,\dfrac{-1\cdot 3\cdot 5}{(2t)^{7/2}}$

. . . .

(47) $p^{-1/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,(2t)^{1/2}$

(48) $p^{-3/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,\dfrac{(2t)^{3/2}}{1\cdot 3}$

(49) $p^{-5/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,\dfrac{(2t)^{5/2}}{1\cdot 3\cdot 5}$

(50) $p^{-7/2} \Big\rceil = \sqrt{\dfrac{2}{\pi}}\,\dfrac{(2t)^{7/2}}{1\cdot 3\cdot 5\cdot 7}$

(51) $\dfrac{p^2}{p+\alpha} \Big\rceil = (p \rceil - \alpha\,\varepsilon^{-\alpha t})$

(52) $\dfrac{p}{p+\alpha} \Big\rceil = \varepsilon^{-\alpha t}$

(53) $\dfrac{1}{p+\alpha} \Big\rceil = \dfrac{1}{\alpha}(1 - \varepsilon^{-\alpha t})$

(54) $\dfrac{1}{p(p+\alpha)} \Big\rceil = \dfrac{t}{\alpha} - \dfrac{1}{\alpha^2} + \dfrac{\varepsilon^{-\alpha t}}{\alpha^2}$

(55) $\dfrac{p^2}{(p+\alpha)(p+\beta)} \Big\rceil = \dfrac{1}{\alpha-\beta}(\alpha\,\varepsilon^{-\alpha t} - \beta\,\varepsilon^{-\beta t})$

(56) $\dfrac{p}{(p+\alpha)(p+\beta)} \Big\rceil = \dfrac{1}{\alpha-\beta}(\varepsilon^{-\beta t} - \varepsilon^{-\alpha t})$

(57) $\dfrac{\omega p}{p^2+\omega^2} \Big\rceil = \sin \omega t$

(58) $\dfrac{p^2}{p^2+\omega^2} \Big\rceil = \cos \omega t$

(59) $\dfrac{\omega^2}{p^2+\omega^2} \Big\rceil = 1 - \cos \omega t$

(60) $\dfrac{\omega p}{p^2-\omega^2} \Big\rceil = \sinh \omega t$

$$(61)\quad \frac{p^2}{p^2-\omega^2}\Bigg| = \cosh \omega t$$

$$(62)\quad \frac{\omega\, p}{(p+\beta)^2+\omega^2}\Bigg| = \varepsilon^{-\beta t} \sin \omega t$$

$$(63)\quad \frac{p\,(p+\beta)}{(p+\beta)^2+\omega^2}\Bigg| = \varepsilon^{-\beta t} \cos \omega t$$

$$(64)\quad \frac{\omega\, p}{(p+\beta)^2-\omega^2}\Bigg| = \varepsilon^{-\beta t} \sinh \omega t$$

$$(65)\quad \frac{p\,\omega \cos\phi \pm p^2 \sin\phi}{p^2+\omega^2}\Bigg| = \sin(\omega t \pm \phi)$$

$$(66)\quad \frac{p^2 \cos\phi \mp \omega\, p \sin\phi}{p^2+\omega^2}\Bigg| = \cos(\omega t \pm \phi)$$

$$(67)\quad \frac{\omega\, p \cos\phi \pm p\,(p+\beta)\sin\phi}{(p+\beta)^2+\omega^2}\Bigg| = \varepsilon^{-\beta t} \sin(\omega t \pm \phi)$$

$$(68)\quad \frac{p\,(p+\beta)\cos\phi \mp \omega\, p \sin\phi}{(p+\beta)^2+\omega^2}\Bigg| = \varepsilon^{-\beta t} \cos(\omega t \pm \phi)$$

In the following three equations, let $\omega^2 = \omega_0^2 - \alpha^2$, $\tan\phi = \omega/\alpha$, and $(-m)$ and $(-n)$ be the two roots of $p^2 + 2\,\alpha\, p + \omega_0^2 = 0$. Then

$$(69)\quad \frac{p^2}{p^2+2\,\alpha\, p+\omega_0^2}\Bigg| = -\frac{\omega_0}{\omega}\varepsilon^{-\alpha t}\sin(\omega t - \phi) \qquad \text{if } \omega_0^2 > \alpha^2$$

$$= \frac{1}{n-m}\left(n\,\varepsilon^{-nt} - m\,\varepsilon^{-mt}\right) \qquad \text{if } \alpha^2 > \omega_0^2$$

$$= \varepsilon^{-\alpha t}(1-\alpha t) \qquad \text{if } \alpha^2 = \omega_0^2$$

$$(70)\quad \frac{p}{p^2+2\,\alpha\, p+\omega_0^2}\Bigg| = \frac{\varepsilon^{-\alpha t}}{\omega}\sin \omega t \qquad \text{if } \omega_0^2 > \alpha^2$$

$$= \frac{1}{n-m}\left(\varepsilon^{-mt} - \varepsilon^{-nt}\right) \qquad \text{if } \alpha^2 > \omega_0^2$$

$$= t\,\varepsilon^{-\alpha t} \qquad \text{if } \alpha^2 = \omega_0^2$$

$$(71)\quad \frac{1}{p^2+2\,\alpha\, p+\omega_0^2}\Bigg| = \frac{1}{\omega_0^2}\left[1 - \frac{\omega_0}{\omega}\varepsilon^{-\alpha t}\sin(\omega t + \phi)\right] \qquad \text{if } \omega_0^2 > \alpha^2$$

$$= \frac{1}{\omega_0^2}\left[1 - \frac{\omega_0^2}{n-m}\left(\frac{\varepsilon^{-mt}}{m} - \frac{\varepsilon^{-nt}}{n}\right)\right] \qquad \text{if } \alpha^2 > \omega_0^2$$

$$= \frac{1}{\omega_0^2}\left[1 - \varepsilon^{-\alpha t}(1+\alpha t)\right] \qquad \text{if } \alpha^2 = \omega_0^2$$

$$(72)\quad \frac{p^2}{(p+\alpha)^2}\Bigg| = \varepsilon^{-\alpha t}(1-\alpha t)$$

$$(73)\quad \frac{p}{(p+\alpha)^2}\Bigg| = t\,\varepsilon^{-\alpha t}$$

(74) $$\frac{1}{(p+\alpha)^2}\;\Bigg|\; = \frac{1}{\alpha^2}\left[1 - \varepsilon^{-\alpha t}(1 + \alpha t)\right]$$

(75) $$\frac{p}{(p^2+\omega^2)(p+\alpha)}\;\Bigg|\; = \frac{1}{\omega\sqrt{\alpha^2+\omega^2}}\left[\varepsilon^{-\alpha t}\sin\beta + \sin(\omega t - \beta)\right]$$

$$= \frac{t^2}{\lfloor\underline{2}} - \frac{t^3}{\lfloor\underline{3}} + \frac{t^4}{\lfloor\underline{4}}(\alpha^2 - \omega^2) - \frac{t^5}{\lfloor\underline{5}}\alpha(\alpha^2 - \omega^2) + \frac{t^6}{\lfloor\underline{6}}(\omega^4 - \alpha^2\omega^2 + \alpha^2)$$

where $\beta = \tan^{-1}\dfrac{\omega}{\alpha}$

(76) $$\frac{p^2 + 2\omega^2}{p^2 + 4\omega^2}\;\Bigg|\; = \cos^2\omega t$$

(77) $$\frac{1}{p^n}\;\Bigg|\; = \frac{t^n}{\lfloor\underline{n}}$$

(78) $$\frac{p}{(p+\alpha)^n}\;\Bigg|\; = \varepsilon^{-\alpha t}\frac{t^{n-1}}{\lfloor\underline{n-1}}$$

(79) $$\frac{(p-\alpha)^2}{(p+\alpha)^2}\;\Bigg|\; = 1 - 4\alpha t\,\varepsilon^{-\alpha t}$$

(80) $$\frac{p}{\sqrt{p^2+\alpha^2}}\;\Bigg|\; = J_0(\alpha t)$$

(81) $$\frac{p}{\sqrt{p^2-\alpha^2}}\;\Bigg|\; = J_0(j\alpha t)$$

(82) $$\frac{p}{\alpha^n\sqrt{p^2+\alpha^2}}\left(\sqrt{p^2+\alpha^2} - p\right)^n\;\Bigg|\; = J_n(\alpha t)$$

(83) $$\varepsilon^{-\frac{\alpha}{p}}\;\Bigg|\; = J_0(2\sqrt{\alpha t})$$

(84) $$\sqrt{\frac{p}{p+2\alpha}} = \varepsilon^{-\alpha t}\frac{p}{\sqrt{p^2-\alpha^2}}\;\Bigg|\; = \varepsilon^{-\alpha t}J_0(j\alpha t)$$

(85) $$\frac{p}{\sqrt{(p+\alpha)^2-\beta^2}}\;\Bigg|\; = \varepsilon^{-\alpha t}J_0(j\beta t)$$

(86) $$\varepsilon^{-\frac{x}{v}\sqrt{(p+\alpha)^2-\beta^2}}\varepsilon^{-\alpha t}J_0(j\beta t)\;\Bigg|\; = \varepsilon^{-\alpha t}J_0\left(j\beta\sqrt{t^2 - \frac{x^2}{v^2}}\right)\Bigg|_{(x/v)}$$

(87) $$\varepsilon^{-\alpha\sqrt{1+p^2}}\frac{p}{\sqrt{1+p^2}}\;\Bigg|\; = J_0\sqrt{t^2-\alpha^2}\;\Bigg|_{(\alpha)}$$

(88) $$\frac{1}{\sqrt{p+\alpha}}\;\Bigg|\; = 2\sqrt{\frac{t}{\pi}}\,\varepsilon^{-\alpha t}\left[1 + \frac{2\alpha t}{1\cdot 3} + \frac{(2\alpha t)^2}{1\cdot 3\cdot 5} + \cdots\right]$$

(89) $\dfrac{p}{\sqrt{p+\alpha}}\Big\rceil = \dfrac{\varepsilon^{-\alpha t}}{\sqrt{\pi t}}$

(90) $\dfrac{p}{\sqrt{p+\alpha}}\,\varepsilon^{-\alpha t}\sqrt{t}\Big\rceil = \dfrac{\sqrt{\pi}}{2}\,\varepsilon^{-\alpha t}\,(1-\alpha t)$

(91) $p\,\varepsilon^{-\alpha t}\Big\rceil = -\alpha\,\varepsilon^{-\alpha t}\Big\rceil + p\Big\rceil$

(92) $p\,f(t)\Big\rceil = f'(t)\Big\rceil + f(o)\,p\Big\rceil$

(93) $p\cos\omega t\Big\rceil = -\omega\sin\omega t\Big\rceil + p\Big\rceil$

(94) $\dfrac{p}{p+\alpha}\,t\,\varepsilon^{-\alpha t}\Big\rceil = \left(1-\dfrac{\alpha t}{2}\right)t\,\varepsilon^{-\alpha t}$

(95) $\varepsilon^{-\alpha t}f(p)\Big\rceil = f(p+\alpha)\,\varepsilon^{-\alpha t}\Big\rceil$

(96) $f(p)\,\varepsilon^{-\alpha t}\Big\rceil = \varepsilon^{-\alpha t}f(p-\alpha)\Big\rceil$

(97) $\sin\omega t\,f(p)\Big\rceil = \dfrac{p}{2j}\left[\dfrac{f(p-j\omega)}{p-j\omega} - \dfrac{f(p+j\omega)}{p+j\omega}\right]\Big\rceil$

(98) $\cos\omega t\,f(p)\Big\rceil = \dfrac{p}{2}\left[\dfrac{f(p-j\omega)}{p-j\omega} + \dfrac{f(p+j\omega)}{p+j\omega}\right]\Big\rceil$

(99) $\varepsilon^{-\alpha p}\Big\rceil = \Big\rceil_{(\alpha)}$

(100) $\varepsilon^{-\alpha p}f(t)\Big\rceil = f(t-\alpha)\Big\rceil_{(\alpha)}$

(101) $p\,\varepsilon^{-\sqrt{\alpha p}}\Big\rceil = \dfrac{1}{2}\sqrt{\dfrac{\alpha}{\pi}}\,\dfrac{\varepsilon^{-\alpha/4t}}{t^{3/2}}$

(102) $\dfrac{1}{1+\sqrt{\alpha/p}}\Big\rceil = \varepsilon^{+\alpha t} - 2\sqrt{\dfrac{\alpha t}{\pi}}\left[1 + \dfrac{2\,\alpha t}{1\cdot 3} + \dfrac{(2\,\alpha t)^2}{1\cdot 3\cdot 5} + \ldots\right]$

(103) $\varepsilon^{-\sqrt{\alpha p}}\Big\rceil = 1 - \sqrt{\dfrac{\alpha}{\pi}}\left[\dfrac{1}{t^{1/2}} - \dfrac{1}{2}\,\dfrac{\alpha}{\lfloor\underline{3}}\,\dfrac{1}{t^{3/2}} + \dfrac{1}{2}\,\dfrac{3}{2}\,\dfrac{\alpha^2}{\lfloor\underline{5}}\,\dfrac{1}{t^{5/2}} + \ldots\right]$

$$= 1 - erf\sqrt{\frac{\alpha}{4t}}$$

(104) $\varepsilon^{\alpha(p-\sqrt{p^2-\beta^2})}\Big\rceil = 1 + T + \dfrac{T^2}{(\lfloor\underline{2})^2} + \left(\dfrac{1}{\lfloor\underline{3}} + \dfrac{1}{\alpha^2\beta^2}\right)\dfrac{T^3}{\lfloor\underline{3}}$

$$+ \left(\frac{1}{\lfloor\underline{4}} + \frac{1}{\alpha^2\beta^2}\right)\frac{T^4}{\lfloor\underline{4}} + \ldots$$

where $T = \alpha\,\beta^2\,t/2$

(105) $\varepsilon^{\alpha(p-\sqrt{p^2-\beta^2})}\dfrac{p}{\sqrt{p^2-\beta^2}} \Big| = 1 + T + \left(\dfrac{1}{\lfloor 2} + \dfrac{\lfloor 2}{\alpha^2\beta^2}\right)\dfrac{T^2}{\lfloor 2}$

$$+ \left(\frac{1}{\lfloor 3} + \frac{\lfloor 3}{\alpha^2\beta^2\lfloor 2}\right)\frac{T^3}{\lfloor 3}$$

$$+ \left(\frac{1}{\lfloor 4} + \frac{3}{\alpha^2\beta^2\lfloor 2} + \frac{\lfloor 3}{\alpha^4\beta^4}\right)\frac{T^4}{\lfloor 4} + \ldots$$

where $T = \dfrac{\alpha\beta^2 t}{2}$

(106) $\varepsilon^{-\frac{x}{v}\sqrt{(p+\alpha)^2-\beta^2}} \Big| = \varepsilon^{-\alpha t}\Bigg[1 + \left(\dfrac{x\beta^2}{2v} + \alpha\right)\left(t - \dfrac{x}{v}\right)$

$$+ \left(\frac{x^2\beta^4}{2^2(\lfloor 2)^2 v^2} + \frac{x\beta^2\alpha}{2^2 v} + \frac{\alpha^2}{\lfloor 2}\right)\left(t - \frac{x}{v}\right)^2 + \ldots\Bigg] 1_{(x/v)}$$

(107) $\dfrac{\sqrt{p}}{p+\alpha}\Big| \equiv \dfrac{1}{\alpha\sqrt{\pi t}}\left[1 + \dfrac{1}{2\alpha t} + \dfrac{1\cdot 3}{(2\alpha t)^2} + \dfrac{1\cdot 3\cdot 5}{(2\alpha t)^3} + \ldots\right]$

asymptotically

(108) $\dfrac{\sqrt{p^3}}{p+\alpha}\Big| \equiv -\dfrac{1}{\sqrt{\pi t}}\left[\dfrac{1}{2\alpha t} + \dfrac{1\cdot 3}{(2\alpha t)^2} + \dfrac{1\cdot 3\cdot 5}{(2\alpha t)^3} + \ldots\right]$

(109) $\dfrac{\sqrt{p}}{1+\sqrt{\alpha p}}\Big| = \dfrac{1}{\sqrt{\alpha}}\varepsilon^{t/\alpha} - \dfrac{2}{\sqrt{\alpha\pi}}\sqrt{\dfrac{t}{\alpha}}\left[1 + \dfrac{2}{3}\left(\dfrac{t}{\alpha}\right) + \dfrac{2^2}{3\cdot 5}\left(\dfrac{t}{\alpha}\right)^2 + \ldots\right]$

$$\equiv \frac{1}{\sqrt{\pi t}}\left[1 - \left(\frac{\alpha}{2t}\right) + 3\left(\frac{\alpha}{2t}\right)^2 - 3\cdot 5\left(\frac{\alpha}{2t}\right)^3 + \ldots\right]$$

asymptotically

(110) $\dfrac{1}{1+\sqrt{(\alpha p)^3}}\Big| = \dfrac{2^2}{3\sqrt{\pi}}\left(\dfrac{t}{\alpha}\right)^{3/2} - \dfrac{1}{\lfloor 3}\left(\dfrac{t}{\alpha}\right)^3 + \dfrac{2^5}{\sqrt{\pi}\,3\cdot 5\cdot 7\cdot 9}$

$$\left(\frac{t}{\alpha}\right)^{9/2} - \frac{1}{\lfloor 6}\left(\frac{t}{\alpha}\right)^6 + \ldots$$

$$\equiv 1 + \frac{1}{2\sqrt{\pi}}\left(\frac{\alpha}{t}\right)^{3/2}\left[1 - 1\cdot 3\cdot 5\cdot 7\left(\frac{\alpha}{2t}\right)^3\right.$$

$$\left. + 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\left(\frac{\alpha}{2t}\right)^6 + \ldots\right]$$

asymptotically

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