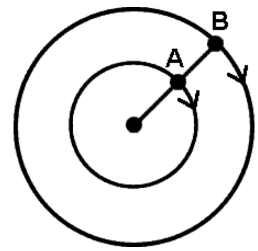
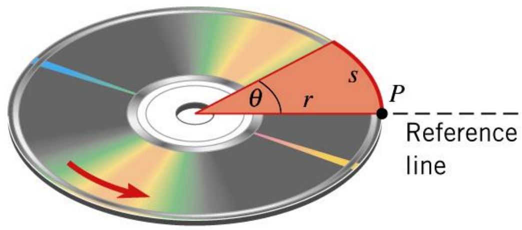
Chapters 8-9 Rotational Kinematics and Dynamics

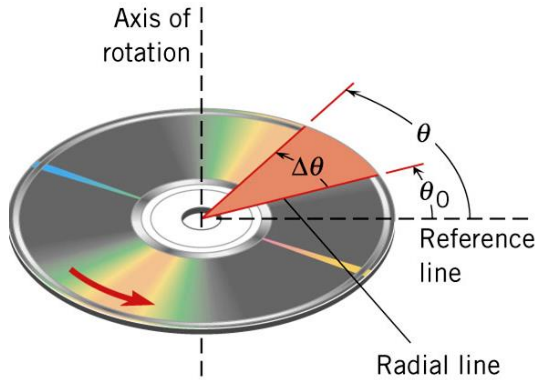
Rotational motion



- **Rotational motion** refers to the motion of an object or system that spins about an axis. The **axis of rotation** is the line about which the rotation occurs. The angle through which an object rotates is its **angular displacement** and is measured in ***radians*** (1 rev = 2π rad = 360°). All points on a rigid object rotating about a fixed axis, except the points on the axis, move through the same angular displacement during the same time interval; therefore, all points have the same angular speed. But the points do ***not*** necessarily have the same linear (tangential) speed. The farther a point is from the axis of rotation, the faster the point is moving. As shown in the diagram to the right, points **A** and **B** have the same angular speed, but point **B** has the larger tangential speed.



Angular displacement in ***radians*** is the ratio of the arc length to radial distance.
$\theta = \Delta s/r$



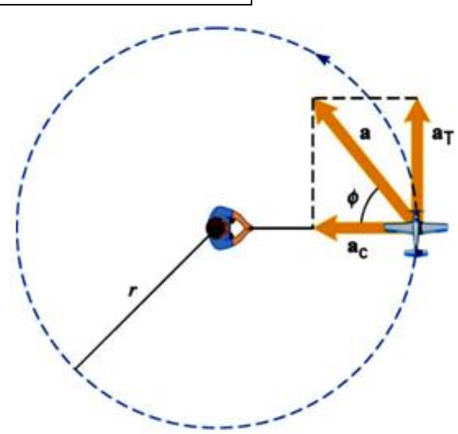
- As shown in the table below, there is a rotational analog for every translational (linear) quantity we have covered. Note that **(1)** angular displacements are measured in radians and if the distance traveled by a point is to be known simply multiple the angular displacement by the radial distance from the axis to the point, **(2)** instead of ***force***, rotational problems use ***torque*** ($\tau \rightarrow$ Greek tau) which not only takes into account the force but also the orientation and the distance the force is from the axis, and **(3)** instead of ***mass***, rotational problems use ***moment of inertia*** (I) which not only takes into account the mass but also how the mass is distributed about the axis of rotation.

Quantity	**Translational**	**Rotational**	**To relate a point**
Displacement	x	$\theta = \Delta s/r$ unit is radian	$x_t = \theta r$
Velocity	v	$\omega = \Delta\theta/\Delta t$	$v_t = \omega r$
Acceleration	a	$\alpha = \Delta\omega/\Delta t$	$a_t = \alpha r$
Kinematics equations	$v = v_o + at$ $x = v_o t + ½at^2$ $v^2 = v_o^2 + 2ax$	$\omega = \omega_o + \alpha t$ $\theta = \omega_o t + ½\alpha t^2$ $\omega^2 = \omega_o^2 + 2\alpha\theta$	
Newton's 1st law	$\Sigma F = 0$	$\Sigma\tau = 0$	
Newton's 2nd law	$\Sigma F = ma$	$\Sigma\tau = I\alpha$	
Work	$W = F \cdot r$	$W = \tau \cdot \theta$	
Kinetic Energy	$K_t = ½\, mv^2$	$K_R = ½\, I\omega^2$	
Momentum	$p = mv$	$L = I\omega$	
Impulse-momentum	$F\Delta t = \Delta p$	$\tau\Delta t = \Delta L$	

Centripetal vs. tangential acceleration

- Centripetal acceleration is experienced by any object undergoing circular motion and is always directed towards the center of the circular path. ***Centripetal acceleration*** changes *direction* of an object, *not* speed. ***Tangential acceleration*** occurs when *speed* of the object is changing and, as its name implies, is tangent to the circular path. *Tangential and centripetal acceleration are perpendicular* (total acceleration can be found using Pythagorean Theorem).

$a_t = \alpha r \qquad a_c = \frac{v_t^2}{r} = \omega^2 r$



Nonuniform circular motion

Difference between mass (translational inertia) and moment of inertia (rotational inertia)

- Moment of inertia is the rotational analog of mass. **Mass** is a measure of the amount of matter of an object or simply the measure of inertia (resistance to changes in ***translational*** motion) of an object. Mass is an intrinsic property of an object that does not change as long as matter is not lost or gained. **Moment of inertia** ($I = \sum mr^2$ with SI units of kg·m^2) is the measure of an objects resistance to changes in ***rotational*** motion. Unlike mass, moment of inertia is ***not*** an intrinsic property of an object, since *moment of inertia not only depends on an object's mass, but also the object's distribution of mass about the axis of rotation.* The farther mass is from the axis of rotation the more the mass contributes to the moment of inertia of the object.

Calculating moment of inertia

- The box below contains formulas for finding the moments of inertia for some common objects. You do not need to memorize any since the formula would be given if it were required for mathematical computations. But for conceptual questions, you do have to know that if two rotating objects have the same mass then the object that has more mass farther from the axis will have a larger moment of inertia. For example, a rolling ring will have a larger moment of inertia than a rolling disk of the same mass and diameter since for the ring all the mass is concentrated at the outer edge.

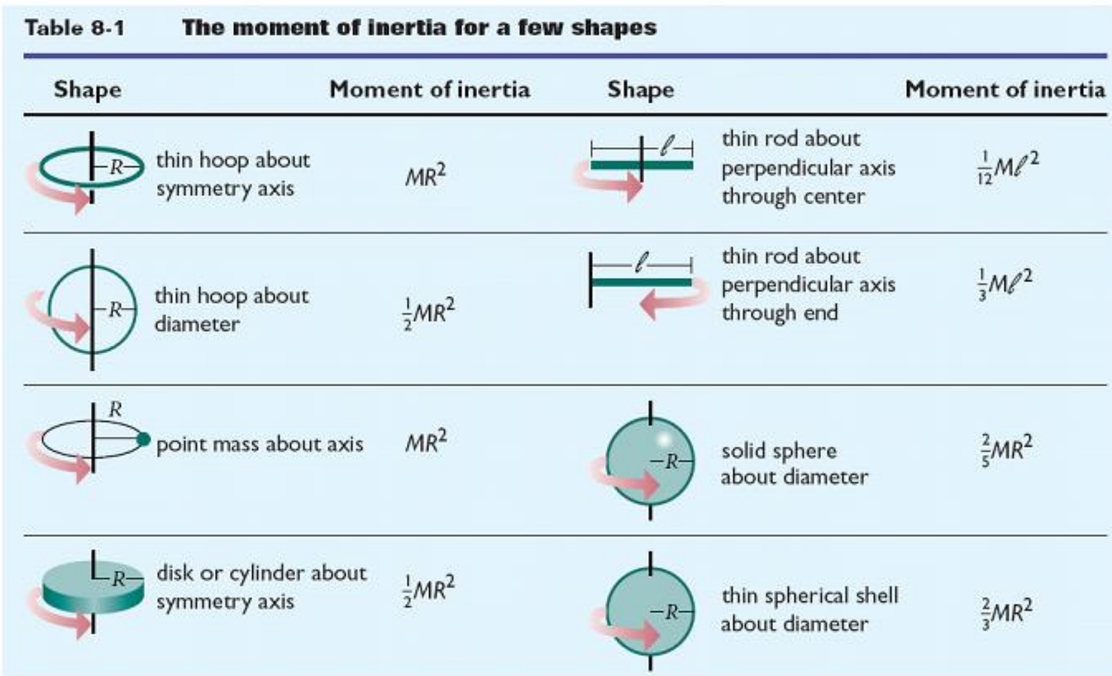
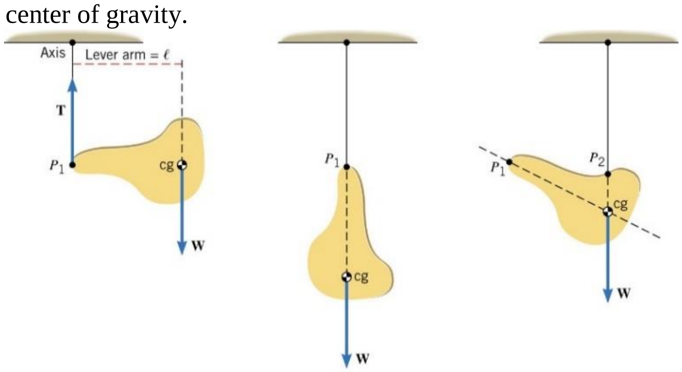


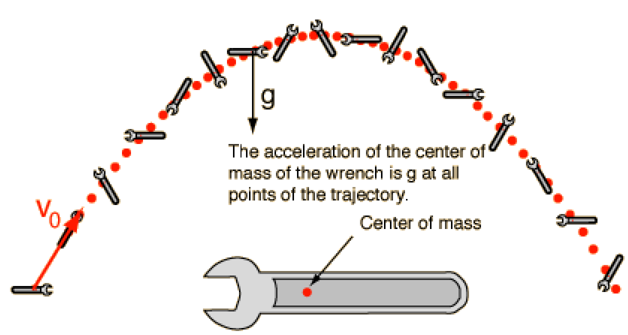
Table 8-1 The moment of inertia for a few shapes

Shape	Moment of inertia	Shape	Moment of inertia
thin hoop about symmetry axis	MR^2	thin rod about perpendicular axis through center	$\frac{1}{12}M\ell^2$
thin hoop about diameter	$\frac{1}{2}MR^2$	thin rod about perpendicular axis through end	$\frac{1}{3}M\ell^2$
point mass about axis	MR^2	solid sphere about diameter	$\frac{2}{5}MR^2$
disk or cylinder about symmetry axis	$\frac{1}{2}MR^2$	thin spherical shell about diameter	$\frac{2}{3}MR^2$

Center of mass and center of gravity

- **Center of mass** is the single point where all the mass of an object can be considered to be concentrated. The **center of gravity** for an object (*located along the vertical line for which the object will balance*) is the single point from which the force of gravity can be considered to act when calculating torque due to the object's weight. For almost all problems, the center of gravity is the same location as the center of mass. The center of gravity for a symmetrical object lies at its geometrical center. As shown below left, the center of gravity for an irregularly shaped object can be located by drawing a vertical line beneath the point of suspension when hung from two different points (the center of gravity is where the two lines intersect). As shown below right, when gravity is the only force acting on a rotating object, it will rotate around its center of gravity.

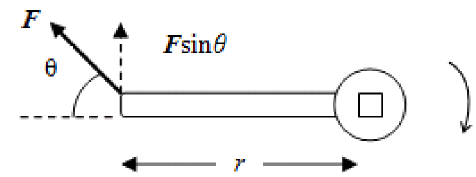




- The center of mass or center of gravity can be calculated by $r_{cm}=\Sigma mr/\Sigma m$ and $r_{cg}=\Sigma mgr/\Sigma mg$ respectively, although you will not need to use those formulas for this class.

Torque

- **Torque** (τ) is a vector quantity that measures the ability of a force to rotate an object about an axis. Torque depends upon the component of force *perpendicular* to the axis of rotation and the radial distance of the force from the axis. Torque is either positive or negative depending on the direction the force tends to rotate the object (*positive if the rotation is counterclockwise and negative if the rotation is clockwise*).



$\tau = r_{\perp}F = Fr\sin\theta$

(SI units are N·m)

Equilibrium revisited

- You should remember from chapter 4 that **equilibrium** describes an object whose motion does not change (*not accelerating*). Even though we did not specify it at that time, equilibrium includes both linear and rotational motion. Sometimes it is necessary to distinguish between translational equilibrium and rotational equilibrium since an object may be in equilibrium for one but not the other. **Translational equilibrium** occurs when the *net force acting on the object is zero* (no linear acceleration). **Rotational equilibrium** occurs when the *net torque acting on the object is zero* (angular acceleration is zero).
- **Static equilibrium** exists when an object *remains at rest* relative to the frame of reference from which it is observed (usually the earth); therefore, static equilibrium requires that the object be in both *translational* and *rotational equilibrium*.

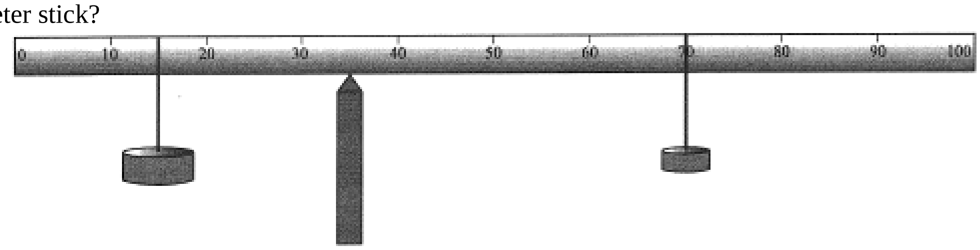
<u>Conditions for Equilibrium</u>

1. Translational equilibrium → $\sum F = 0$
2. Rotational equilibrium → $\sum \tau = 0$
3. Equilibrium → $\sum F = 0$ and $\sum \tau = 0$
4. Static equilibrium → remains at rest; $\sum F = 0$ and $\sum \tau = 0$

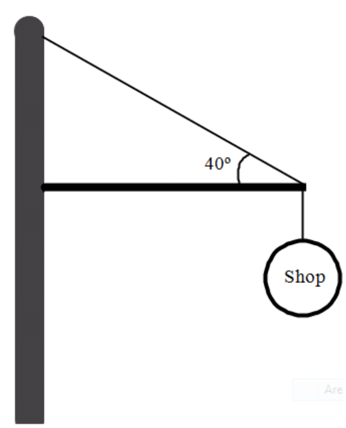
<u>Steps for solving static equilibrium problems</u>

1. Draw a free-body diagram with the object shown as a likeness of the object (***not just a dot***). Make sure each force vector is drawn from the point of application of the force (center of gravity for weight).
2. Pick an axis or pivot about which the torques will be calculated. One nice thing about rotational equilibrium problems is that *you can choose whatever point you want to be the axis of rotation*, allowing you to pick a point with an unknown variable in order to eliminate it.
3. Determine the torque about the chosen pivot point due to each force. Make sure to take into account direction (positive if counterclockwise and negative if clockwise).
4. Write equations for $\sum \tau = 0$, $\sum F_x = 0$, and $\sum F_y = 0$. Solve for all unknown variables.

Example 1: The uniform meter stick above has an object with mass 800 grams hanging at the 15 cm mark and an object with mass 350 grams at the 70 cm mark. It balances horizontally on a pivot placed at the 35 cm mark. What is the mass of the meter stick?



Example 2: A uniform metal beam with a mass of 20 kg extends horizontally from a pole, as shown right. A support cable of negligible mass extends from the far end of the beam to the pole, forming a 40º angle with the beam. The beam has a sign with a mass of 5 kg hanging from the end of it.



(a) Find the tension in the cable that helps support the beam and the sign.

(b) Find the horizontal and vertical components of the force the pole exerts on the metal beam.

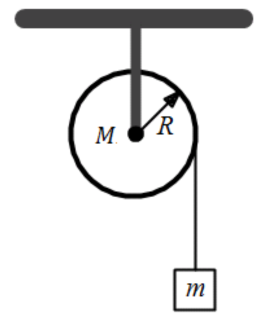
Net torque and angular acceleration

- When a net torque is applied to an object, the object will have an angular acceleration that is directly proportional to the net torque and inversely proportional to the moment of inertia ($\sum \tau = I\alpha$). This is Newton's 2nd Law for rotational motion. Mathematical problems will typically include rigid bodies (so moment of inertia remains constant) that rotate about a fixed axis (to avoid complications that arise for an object undergoing a combination of rotational and translational motion). The problem solving strategy for rotational dynamics is very similar to that of linear dynamics covered in Chapter 4.

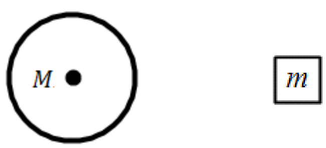
<u>**Steps for solving rotational dynamics problems**</u>

1. Draw a free-body diagram with the object shown as a likeness of the object (***not just a dot***). Make sure each force vector is drawn from the point of application of the force.
2. Identify the axis about which the object rotates and determine the torque (including proper sign) due to each force.
3. Write an equation for $\sum \tau = I\alpha$.
4. If necessary, write additional equations for both the *x* and *y* axis for the object's translational motion ($\sum F = ma$ or $\sum F = 0$) until there are enough equations to solve for all unknown variables. Remember that linear and angular quantities can be related by multiplying the angular quantity by the radial distance to the point of the linear quantity ($a_t = \alpha r$).

Example 3: A uniform solid cylinder of mass M = 1.5 kg and radius R = 0.25 m is mounted on frictionless bearings about a fixed axis. The rotational inertia of the cylinder about its axis is $I = \frac{1}{2}MR^2$. A block of mass m = 75 g is suspended by a cord wrapped around the cylinder and released at time t = 0.



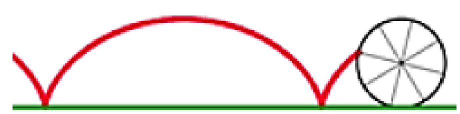
(a) On the diagrams below, draw and identify all of the forces acting on the cylinder and on the block.

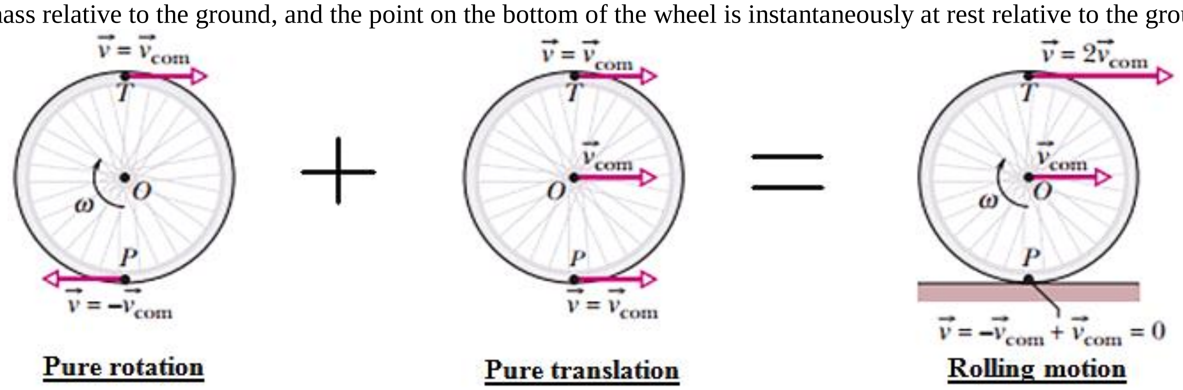


(b) Determine the magnitude of both the acceleration of the block and the tension in the cord.

Rolling without slipping

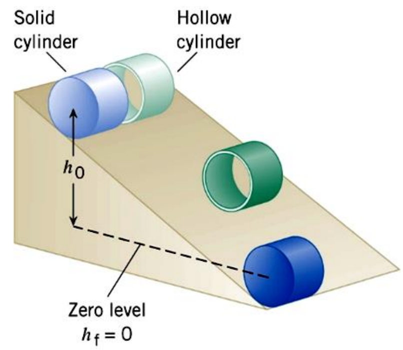
- An object rolling without slipping is a combination of both rotational and translational motion. As shown right, tracing a point on the edge of a circular wheel would show that the path of the point relative to the ground is a ***cycloid***. As shown below, the velocity of the wheel's center of mass relative to the *ground* is equal to the velocity of a point on the wheel's edge relative to the *axis*. The velocity of any point on the wheel relative to the ground can be found by the vector sum of the velocity at that point when the wheel is in pure rotation to the velocity of the point when the wheel is in pure translation. Thus the speed of the top part of the wheel relative to the ground is twice the speed of the wheel's center of mass relative to the ground, and the point on the bottom of the wheel is instantaneously at rest relative to the ground.





Rotational kinetic energy and conservation of mechanical energy under nonslip conditions

- Spinning objects have ***rotational kinetic energy*** ($K_R = \frac{1}{2}I\omega^2$) which is a separate quantity from translational kinetic energy. Changes in rotational kinetic energy must be considered when analyzing conservation of mechanical energy ($ME = \sum K + \sum U$). Look at the diagram to the right. If a solid disk and a thin ring with the same mass and diameter start from rest at the top of an inclined plane at the same time, and both roll down the incline without slipping, the disk will reach the bottom first and with the fastest linear speed. This is because the disk has a smaller moment of inertia. As the cylinders roll down, potential energy is converted into kinetic energy, but the kinetic energy is shared between the translational and rotational forms. The object with more of its kinetic energy in the translational form (disk) will have a greater linear speed. What if each cylinder was rolling with the *same speed* as they approached the incline? Which would travel farther up the incline assuming both roll without slipping? The ring would since it has more total kinetic energy at the bottom of the incline.



Example 4: A bowling ball with uniform density has a mass of 5.0 kg and a radius of 0.12 m. The ball is rolling horizontally without slipping with a linear speed of 2.0 m/s when the ball encounters an incline. ($I_{solid\ sphere\ about\ diameter} = \frac{2}{5}MR^2$)

(a) If the ball rolls without slipping up the incline, what maximum vertical height will the ball reach?

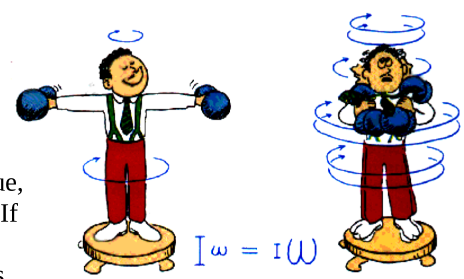
(b) Suppose, instead, that the ball were to roll toward the incline as stated above, but the incline were frictionless. State whether the height reached by the ball would be less than, greater than, or equal to the height calculated in (a).

_____Less than   _____Greater than   _____Equal to

Explain briefly.

Conservation of angular momentum

- Rotating objects possess ***angular momentum*** ($L = I\omega$) which is the product of the object's moment of inertia and angular velocity. You should remember that linear momentum is always conserved as long as *no net external forces act on the system* ($\sum F_{external} = 0$). Angular momentum is always conserved as long as *no net external torques act on the system* ($\sum \tau_{external} = 0$). This means that with no net external torque, the product of rotational velocity and angular velocity remain constant ($I_0\omega_0 = I_f\omega_f$). If moment of inertia increases, angular speed decreases. If moment of inertia decreases, angular speed increases. This is why the spinning skater speeds up as he pulls his arms inward. Angular momentum is conserved, so as his moment of inertia decreases is angular speed increases. But wait, this means his rotational kinetic energy increases. Where did the energy come from? He did work in moving his arms.



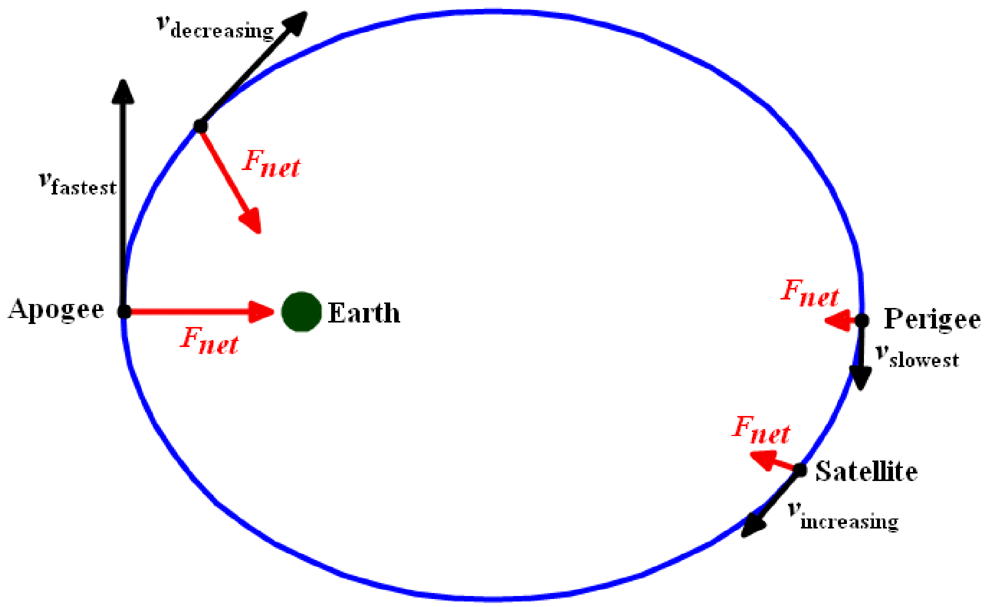
Example 5: A figure skater goes into a spin with arms fully extended. Complete the following table for the changes that occur, if any, as the skater pulls her arms inward.

Quantity	Increases	Decreases	Remains constant
Angular momentum			
Moment of inertia			
Angular speed			
Rotational kinetic energy			

If the moment of inertia of the skater with arms fully extended is 4.8 kg·m^2 and her angular speed is 4.0 rad/s, what will be her angular speed after her arms are pulled inward if this reduces her moment of inertia to 3.6 kg·m^2?

Conservation of angular momentum and elliptical orbits

- As shown below, the speed of a satellite in an elliptical orbit around earth changes from a maximum at closest approach (***apogee***) to a minimum when the satellite is at its farthest distance from the earth (***perigee***). As the satellite moves from the apogee to the perigee, there is a component of the net force (gravity) that slows the satellite. As the satellite moves from the perigee to the apogee, there is a component of the net force that speeds up the satellite. However, at any instant, the gravitational force is directed toward the center of the earth and passes through the axis about which the satellite instantaneously rotates. Therefore, *the gravitational force exerts no torque on the satellite*. Consequently, *the angular momentum of the satellite remains constant at all times*. Therefore, as the moment of inertia of the satellite decreases (gets closer to earth), the speed of the satellite increases. The reverse is true as the satellite moves away from the earth (from the apogee to the perigee). ***Note*** that mechanical energy of the satellite is also conserved. As the satellite falls toward the earth, the potential energy ($U = -G\frac{m_1m_2}{r}$) decreases and the kinetic energy ($K = \frac{1}{2}mv^2$) increases by the same amount, with the opposite occurring as the satellite moves away from the earth.



Conservation of linear and angular momentum

- Many problems involve both conservation of linear and angular momentum. Remember from chapter 7 the problem with the astronaut returning to her spaceship by throwing a wrench in the opposite direction of the spaceship. If the astronaut does ***not*** throw the wrench along a radial line passing through her center of mass, not only will she move in the opposite direction from the wrench, but she will also spin. This is because both a net force and a net torque will be applied to her by the wrench. The net torque depends on the component of the force she applies to the wrench that is perpendicular to the radial line passing through her center of mass. Once the wrench is thrown, she will rotate around her center of mass (with the axis of rotation depending on the direction of the net torque) and move linearly in the opposite direction of the wrench.

Example 6: A space shuttle astronaut in a circular orbit around the Earth has an assembly consisting of two small dense spheres, each of mass *m*, whose centers are connected by a rigid rod of length ℓ and negligible mass. The astronaut also has a device that will launch a small lump of clay of mass *m* at speed v_0. The assembly is "floating" freely at rest relative to the cabin when the astronaut launches the clay lump so that it perpendicularly strikes and sticks to one of the spheres of the assembly, as shown right. Describe the subsequent motion of the system after the collision.