Electromagnetic and gravitational waves in a stationary magnetic field

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The interaction between electromagnetic and gravitational waves in an external stationary magnetic field is investigated. In the strictly coherent case, the transformation of one wave into another is complete. (A particular case of this process is the effect discovered by Gertsenshteĭn.) In the presence of matter, refraction of electromagnetic waves reduces the coherence length and the effect practically disappears.

In the remarkable paper of Gertsenshteĭn[1] he considered the transformation of an electromagnetic wave (EMW) into a gravitational wave when the electromagnetic wave propagates through a constant transverse magnetic field H_0. The EMW is transformed into a GW of the same frequency and wave vector, due to the equality of their velocities of propagation. Thus, in[1] the role of coherence in the transformation under consideration was exhibited.

The reverse process GW $\rightarrow$ EMW in a magnetic field was considered in a number of papers[2-4]. When an EMW (E, H) propagates in the field H_0 there appears a stress tensor proportional to HH_0 which is variable in space and time. This tensor is the source of GW. When a GW propagates through the field H_0 there occurs a stretching and compression of the magnetic field, accompanied by the appearance of an alternating magnetic field $h(x, t)H_0$, where h is the variation of the metric in the GW. The field hH_0 is the source for the EMW. We single out especially the paper of the Italian authors[3]. Starting from an understanding of the role of coherence these authors noted the influence of the medium on the EMW, since this influence destroys the coherence and limits the conversion GW $\rightarrow$ EMW; it is obvious that a medium also affects the inverse process.

In the present note both processes EMW $\rightleftarrows$ GW are considered together in a unified manner, which allows one to break out of the frame of the small-conversion approximation. We first consider the idealized coherent case. In this case the conversion has an oscillatory character: 100% EMW $\rightarrow$ 100% GW $\rightarrow$ 100% EMW; however, the whole cycle requires a length $X_0 \approx c^2/H_0G^{1/2} \approx 10^{24}/H_0$ (here X_0 is in cm, H_0 in Oe) distance which is huge even on an astrophysical scale.

A rigorous discussion of waves in a magnetic field H_0 leads to the introduction of normal modes which are mixed (EMW, GW) with different phase relations.

Further we consider systematically those factors which violate the coherence. Some of them are related to the medium—the atoms, electrons, ions and neutrinos which exist in the space in which the conversion is considered. But also in vacuum the nonlinearity of electrodynamics (at short wavelengths) and the influence of H_0 on the GW itself (for ultralong waves) sets limits to the coherence. The dispersion equation becomes more complicated. In astrophysical situations the effects are small even under extreme assumptions on magnetic fields in a pulsar, or on an ordered cosmological magnetic field.

Before discussing the more rigorous theory we give a few estimates for astrophysical conditions, based on the "coherent" result, appropriated from[1]. The fraction of energy of the EMW transformed into the energy of GW in the field H_0 along the pathlength R equals

$$\alpha = GH_0^2R^2/c^4. \tag{1}$$

This quantity is small under laboratory conditions and even under pulsar conditions:

$H_0=10^5\,\text{Oe}$, $R=10^3\,\text{cm}$, $\alpha=10^{-33}$; $H_0=10^{13}\,\text{Oe}$, $R=10^6\,\text{cm}$, $\alpha=10^{-11}$.

In a universe with a homogeneous magnetic field varying according to the freezing-in law $H = H_0(1 + z)^2$, where H_0 is today's field, z is the redshift, the effect could be substantial. Let us express the time in terms of z: $dt = \mathscr{H}^{-1}(1 + z)^{-5/2}dz$, where $\mathscr{H}$ is the present day Hubble constant. We generalize α to the case of a variable field:

$$\alpha = Gc^{-2}\left[\int H(t)\,dt\right]^2 = {}^1/_4 Gc^{-2}\mathscr{H}^{-2}H_0^2[(1+z)^{1/2}-1]^2 \approx {}^1/_4 zGc^{-2}\mathscr{H}^{-2}H_0^2, \quad z\gg 1. \tag{2}$$

Let us set $\mathscr{H} = 50$ km/s-Mpc $= 1.6 \times 10^{-18}\,\text{s}^{-1}$, $H_0 = 3 \times 10^{-6}$ Oe. This field is the upper limit selected according to the condition $H_0^2/8\pi = \epsilon_r$, where ϵ_r is the energy density of the T = 2.7 K microwave background radiation. In reality a cosmological field—should it exist—will not exceed 10^{-8} Oe. Still, taking the value 3×10^{-6} we obtain $\alpha = 10^{-4}$ z. Thus, for $z \sim 10^3$ we obtain $\alpha \sim 0.1$. This would mean a reduction by 10% of the intensity of the background radiation in a wide belt perpendicular to the cosmological field; moreover the suppression would occur for radiation of a definite polarization. The experiments seem to indicate that such effects do not exist even at a level smaller by a factor 1000.[1)]

However, we shall see below that a breakdown of coherence reduces the effect to a magnitude $\alpha < 10^{-12}$. Therefore the process EMW $\rightleftarrows$ GW does not allow one to draw any conclusions on the existence of a cosmologic magnetic field.

We pose the academic question on how one should consider the situation with $\alpha > 1$. It is obvious that in addition to the conversion of EMW into GW the reverse process also occurs: in thermodynamic equilibrium the energy of the EMW and of the GW is the same. An exact discussion of the coupled equations for the EMW and the GW obviously leads to the concept of two normal modes. We consider equations of the type $\square a = pb$ and $\square b = pa$, where a refers to the EMW, b refers to the GW, such that the energy density $\epsilon = ka^2 = kb^2$[2] has the same expression in terms of a and b^2 (e.g., $a = H/\sqrt{8\pi}$, $b \sim \dot{h}/\sqrt{G}$). Then the normal modes consist of superpositions of EMW and GW:

$$f=\frac{1}{\sqrt{2}}\left\{\begin{matrix}a=e^{ikx-i\omega t}\\ b=e^{ikx-i\omega t}\end{matrix}\right\},\quad g=\frac{1}{\sqrt{2}}\left\{\begin{matrix}a=e^{ikx-i\omega t}\\ b=-e^{ikx-i\omega t}\end{matrix}\right\}. \quad (3)$$

The equations for f and g have the form

$$\Box f=pf,\quad \Box g=-pg. \quad (4)$$

The difference between the phase velocities of the normal modes is substantial:

$$\omega_f=c\sqrt{k^2-p}\approx c(k-p/2k),\quad \omega_g\approx c(k+p/2k). \quad (5)$$

When it enters the region occupied by the constant field, a pure EMW should be considered as a superposition of f- and g-waves.

As the f-and g-waves of the same ω (equal to the ω of the EMW incident upon the region) propagate, the k values for these waves differ. Their phaseshift during the propagation, as x increases, signifies a partial conversion of the EMW into a GW:

$$\{a,b\}=\frac{1}{\sqrt{2}}e^{-i\omega t}(fe^{ik_fx}+ge^{ik_gx})=\exp\left(-i\omega t+i\frac{k_f+k_g}{2}x\right)$$
$$\times\left[\{a\}\cos\frac{(k_f-k_g)x}{2}+\{b\}\sin\frac{(k_f-k_g)x}{2}\right]. \quad (6)$$

For a small pathlength this equation yields a quadratic law of increase of the energy converted into the GW:

$$\alpha=\left(\frac{k_f-k_g}{2}\right)^2x^2=\frac{c^2p^2}{4\omega^2}x^2. \quad (7)$$

Comparing (7) with Gertsenshteĭn's formula we obtain[3)]

$$p=\frac{2\omega H_0}{c^3}\sqrt{G}\approx\frac{2kH_0}{c^2}\sqrt{G}. \quad (8)$$

Here the fractional power $G^{1/2}$ is obtained because the GW is described by the amplitude b normalized in such a manner that the energy density of the GW is proportional to b^2 without the factor G:

$$\varepsilon=kb^2\sim\frac{k}{G}\hbar^2,\quad b\sim\frac{\hbar}{\sqrt{G}}.$$

Along a path corresponding to $\alpha > 1$ there occur the successive conversions EMW → GW → EMW → GW...

It is necessary to take into account perturbations of other kinds which reduce the coherence length and therefore limit and make incomplete the conversion of EMW into GW and vice versa.

For EMW it is necessary to take into account: 1) the change of the speed of light corresponding to the quantum-electrodynamic correction αH^4 to the Lagrangian (Heisenberg and Euler[6], Schwinger[7]); 2) the index of refraction of the medium, depending on the presence of atoms and free charges.

The equation for the GW also contains a correction corresponding to the deviation of the index of refraction from one in the presence of a constant magnetic field. However, this effect is small, of the order p/k^2, where p is the coupling coefficient of the GW and the EMW (cf. Eq. (8)), and k is the propagation vector. In addition, one must take into account the contribution of the photons of the background radiation, which can be considered as an ultrarelativistic collisionless gas at the frequencies which interest us (Bashkov[8], Polnarev[9]).

Thus, we obtain

$$\Box a=pb+ra,\quad \Box b=pa+qb. \quad (9)$$

The order of magnitude is

$$q=\frac{G(H^2+\gamma\varepsilon_r)}{c^4},\quad r=\left(\frac{e^2}{\hbar c}\right)^2\frac{\hbar^3H^2}{m^4c^5}k^2+n_ar_0{}^3k^2-\frac{\omega_0{}^2}{c^2}. \quad (10)$$

Here r_0 is the Bohr radius, r_0^3 being the order of magnitude of the polarizability of the atom; $\omega_0^2 = n_e e^2/m$; n_a is the density of atoms, n_e is the density of electrons; the quantity γ is of the order 1.

Apparently r also contains a term of order q, which is proportional to the gravitational constant G; we neglect it, as well as q itself. For q = 0 the system (9) yields two modes with the shift-coefficient p/r (for $r \gg p$)

$$f'=a+\frac{p}{r}b,\quad g'=b-\frac{p}{r}a. \quad (11)$$

Under cosmological conditions $z = 1000$, $H = 1$, $n_a = 10^3$, $n_e = 10^{-1}$, $k = 10^4$, we obtain the following estimates:

$$p=10^{-20}\ \mathrm{cm}^{-2},$$
$$r_1=\left(\frac{e^2}{\hbar c}\right)^2\frac{\hbar^3H^2}{m^4c^5}k^2\approx10^{-22}\ \mathrm{cm}^{-2},\quad r_2=n_ar_0{}^3k^2\approx10^{-14}\ \mathrm{cm}^{-2},$$
$$r_3=n\frac{e^2}{mc^2}\approx10^{-15}\ \mathrm{cm}^{-2},\quad r\sim10^{-14}\ \mathrm{cm}^{-2},\quad \frac{p}{r}\sim10^{-6}.$$

The neglected quantity q is of the order $10^{-48}\ \mathrm{cm}^{-2}$. Under these conditions the refraction of the EMW limits the coherence length of the EMW and the GW to a quantity $L = k/r = 10^{18}$ cm, and the conversion coefficient is $\alpha_{max} = (p/r)^2 = 10^{-12}$.

Thus, a considerable conversion coefficient of the EMW into GW would be possible only in an empty ($n_a = n_e = 0$) hot magnetic Universe where one can also realize large z, up to 10^8. In the presence of matter the effect disappears for all practical purposes.

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1) A decrease of intensity is obtained if one assumes that at an early stage there was thermodynamic equilibrium between the GW and the EMW. In the course of the cooling-down particles with nonzero restmass disappear; their energy transfers to EMW and therefore at a later stage, for $z < 10^9$, the temperature of the EMW is several times larger than the temperature of the GW, and therefore the coupling between the GW and the EMW leads to a lowering of the temperature of the EMW.

2) The quantities a and b represent tensors; we can omit the tensor indices since the exact theory was developed in the preceding papers[1-3].

3) We note that the dispersion law (5) for the case when p is proportional to k yields, according to (8), a group velocity for both waves (f and g) equal to the speed of light. It is curious that at p = const > 0 Eq. (5) yields a group velocity for the f-wave which exceeds the speed of light, $\partial\omega_f/\partial k = c[1 + (p/2k^2)]$. However, this does not mean that causality is violated: the characteristics of the equation for f still propagate with the speed of light, and it is impossible to transmit information with superluminous velocities[5]. In reality the physical singularity of the equation for f is an instability: the presence of solutions which increase exponentially with time for k = 0 or $|k| < p^{1/2}$. Substituting p according to (8) we obtain the critical value $|k|_c = 2H_0G^{1/2}/c^2$ and the corresponding wavelength $\lambda\!\!\!^{-}_c = k_c^{-1}$. If the size of the region occupied by the magnetic field is smaller than $\lambda\!\!\!^{-}_c$, the instability does not manifest itself. The wavelength $\lambda\!\!\!^{-}_c$ corresponds just to the distance of complete conversion of a GW into an EMW or vice versa.

On the other hand over a dimension of the order $\lambda\!\!\!^{-}_c$ of the region occupied by the field the gravitational potential of the magnetic field itself is of the order c^2, the variation of the metric is of the order of unity, loss of stability by the field, accompanied by collapse, is close. It is obvious that a simple discussion is limited by the condition $R < \lambda\!\!\!^{-}_c$.

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